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ON THE FAN AND LIDSKII MAJORISATIONS OF POSITIVE SEMIDEFINITE MATRICES

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Abstract. Let A,B be two positive definite matrices. The Fan-Lidskii majorisations can be subsumed as the symmetric norm rearrangement inequalities

$$\|A^{\downarrow}+B^{\uparrow}\| \leqslant \|A+B\| \leqslant \|A^{\downarrow}+B^{\downarrow}\|$$

where the up/down arrows on A,B mean the diagonal matrices with the same eigenvalues in decreasing/increasing order down to the diagonal. We refine these relations with sums of type $A+UBU^*$ and $A+VBV^*$ for two unitary matrices U,V associated in a quite natural way to A and B. Stronger results than majorisation are given by using simple averages in the unitary orbits of $A+B$, $A+UBU^*$ and $A+VBV^*$. Proofs rely on an orbital decomposition for positive block-matrices.

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