

STARLIKENESS OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

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Abstract. In this paper necessary and sufficient conditions are obtained for the starlikeness of Bessel functions of the first kind and their derivatives of the second and third order by using a result of Shah and Trimble about transcendental entire functions with univalent derivatives and Mittag-Leffler expansions for the derivatives of Bessel functions of the first kind, as well as some results on the zeros of these functions.

Mathematics subject classification (2010): 33C10, 30C45.

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