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# ON ONE EXTENSION THEOREM DEALING WITH WEIGHTED ORLICZ–SLOBODETSKII SPACE. ANALYSIS ON LIPSCHITZ SUBGRAPH AND LIPSCHITZ DOMAIN

RAJ NARAYAN DHARA AND AGNIESZKA KAŁAMAJSKA

*Abstract.* Having a given weight $\rho(x) = \tau(\operatorname{dist}(x, \partial\Omega))$ defined on Lipschitz boundary domain $\Omega$ and an Orlicz function $\Psi$, we construct the subordinated weight $\omega(\cdot,\cdot)$ defined on $\partial\Omega \times \partial\Omega$ and extension operator $\operatorname{Ext}^L : Lip(\partial\Omega) \mapsto Lip(\overline{\Omega})$ form Lipschitz functions defined on $\partial\Omega$ to Lipschitz functions defined on $\overline{\Omega}$, independent of $\tau$ and $\Psi$, in such a way that $\operatorname{Ext}^L$ extends to the bounded operator from the subspace of weighted Orlicz-Slobodetskii space $Y_\omega^{\Psi,\Psi}(\partial\Omega)$ generated by Lipschitz functions and subordinated to the weight $\omega$ to Orlicz-Sobolev space $W_\rho^{1,\Psi}(\Omega)$. More detailed analysis on Lipschitz subgraph is also provided. Result is new in the unweighted Orlicz setting for general function $\Psi$ as well as in the weighted $L^p$ setting.

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