

## INEQUALITIES FOR AVERAGES OF QUASICONVEX AND SUPERQUADRATIC FUNCTIONS

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*Abstract.* For  $n \in \mathbb{Z}_+$  we consider the difference

$$B_{n-1}(f) - B_n(f) := \frac{1}{a_n} \sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}} \sum_{i=0}^n f\left(\frac{a_i}{a_n}\right)$$

where the sequences  $\{a_i\}$  and  $\{a_i - a_{i-1}\}$  are increasing. Some lower bounds are derived when  $f$  is 1-quasiconvex and when  $f$  is a closely related superquadratic function. In particular, by using some fairly new results concerning the so called "Jensen gap", these bounds can be compared. Some applications and related results about

$$A_{n+1}(f) - A_n(f) := \frac{1}{a_n} \sum_{i=1}^n f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right)$$

are also included.

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