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SHARP BOUNDS FOR TOADER–QI MEAN IN TERMS OF LOGARITHMIC AND IDENTRIC MEANS

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Abstract. In the article, we prove that the double inequality $\lambda\sqrt{L(a,b)I(a,b)} < TQ(a,b) < \mu\sqrt{L(a,b)I(a,b)}$ holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda \leqslant \sqrt{e/\pi}$ and $\mu \geqslant 1$, and give an affirmative answer to the conjecture proposed by Yang in [39], where $L(a,b) = (b-a)/(\log b - \log a)$, $I(a,b) = (b^b/a^a)^{1/(b-a)}/e$ and $TQ(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta$ are respectively the logarithmic, identric and Toader-Qi means of a and b.

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