CHARACTERIZATION OF THE HARDY PROPERTY
OF MEANS AND THE BEST HARDY CONSTANTS

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Abstract. The aim of this paper is to characterize in broad classes of means the so-called Hardy means, i.e., those means $M \colon \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \to \mathbb{R}_+$ that satisfy the inequality

$$\sum_{n=1}^{\infty} M(x_1, \ldots, x_n) \leqslant C \sum_{n=1}^{\infty} x_n$$

for all positive sequences (x_n) with some finite positive constant C. One of the main results offers a characterization of Hardy means in the class of symmetric, increasing, Jensen concave and repetition invariant means and also a formula for the best constant C satisfying the above inequality.

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