TURÁN TYPE OSCILLATION INEQUALITIES IN L^q NORM ON THE BOUNDARY OF CONVEX DOMAINS

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Abstract. Some 77 years ago P. Turán was the first to establish lower estimations of the ratio of the maximum norm of the derivatives of polynomials and the maximum norm of the polynomials themselves on the interval $\mathbb{I} := [-1,1]$ and on the unit disk $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| \leqslant 1\}$ under the normalization condition that the zeroes of the polynomial p all lie in the interval or in the disk, respectively. He proved that with $n := \deg p$ tending to infinity, the precise growth order of the minimal possible ratio of the derivative norm and the norm is $\sqrt{n}$ for $\mathbb{I}$ and n for $\mathbb{D}$.

J. Erőd continued the work of Turán and extended his results to several other domains. The growth of the minimal possible ratio of the ∞-norm of the derivative and the polynomial itself was proved to be of order n for all compact convex domains a decade ago.

Although Turán himself gave comments about the above oscillation question in L^q norms, till recently results were known only for $\mathbb{D}$ and $\mathbb{I}$. Here we prove that in L^q norm the oscillation order is again n for a certain class of convex domains, including all smooth convex domains and also convex polygonal domains having no acute angles at their vertices.

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