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# REMARKS ON SOME NORM INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES AND QUESTIONS OF BOURIN

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*Abstract.* Let $A$ and $B$ be positive semidefinite matrices. It is shown that

$$\left\|A^sB^p+B^qA^t\right\|_2 \leqslant \left\|A^sB^p+A^tB^q\right\|_2$$

for all positive real numbers $s,t,p,q$ for which

$$\left|\frac{s}{s+t}-\frac{1}{2}\right|+\left|\frac{p}{p+q}-\frac{1}{2}\right| \leqslant \frac{1}{2}.$$

This is a generalization of a recent inequality proved by Bhatia for the special case $s=q$, $t=p$ with

$$\frac{1}{4} \leqslant \frac{p}{p+q} \leqslant \frac{3}{4},$$

and it is a special case of a conjecture posed by Hayajneh and Kittaneh, which claims that for positive semidefinite matrices $A_1,A_2,B_1,B_2$ with $A_1B_1=B_1A_1$, $A_2B_2=B_2A_2$ and any unitarily invariant norm,

$$|||A_1B_2+A_2B_1||| \leqslant |||A_1B_2+B_1A_2|||.$$

For $i=1,\ldots,k$, let $A_i$ and $B_i$ be positive semidefinite matrices such that, for each $i$, $A_i$ commutes with $B_i$. It is shown that for any unitarily invariant norm,

$$\left|\left|\left|\sum_{i=1}^k A_iB_i\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^k A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^k B_i\right)\left(\sum_{i=1}^k A_i\right)^{\frac{1}{2}}\right|\right|\right|.$$

This is stronger than the inequality

$$\left|\left|\left|\sum_{i=1}^k A_iB_i\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^k A_i\right)\left(\sum_{i=1}^k B_i\right)\right|\right|\right|,$$

which has been recently proved by Audenaert. Simple applications of these norm inequalities answer some questions of Bourin affirmatively.

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