REMARKS ON SOME NORM INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES AND QUESTIONS OF BOURIN

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Abstract. Let $A$ and $B$ be positive semidefinite matrices. It is shown that
$$
\|A^s B^p + B^q A^t\|_2 \leq \|A^s B^p + A^t B^q\|_2
$$
for all positive real numbers $s, t, p, q$ for which
$$
\left|\frac{s}{s+t} - \frac{1}{2}\right| + \left|\frac{p}{p+q} - \frac{1}{2}\right| \leq \frac{1}{2}.
$$
This is a generalization of a recent inequality proved by Bhatia for the special case $s = q$, $t = p$ with
$$
\frac{1}{4} \leq \frac{p}{p+q} \leq \frac{3}{4},
$$
and it is a special case of a conjecture posed by Hayajneh and Kittaneh, which claims that for positive semidefinite matrices $A_1, A_2, B_1, B_2$ with $A_1 B_1 = B_1 A_1$, $A_2 B_2 = B_2 A_2$ and any unitarily invariant norm,
$$
\|A_1 B_2 + A_2 B_1\| \leq \|A_1 B_2 + B_1 A_2\|.
$$

For $i = 1, \ldots, k$, let $A_i$ and $B_i$ be positive semidefinite matrices such that, for each $i$, $A_i$ commutes with $B_i$. It is shown that for any unitarily invariant norm,
$$
\left\| \sum_{i=1}^k A_i B_i \right\| \leq \left\| \left( \sum_{i=1}^k A_i \right)^{\frac{1}{2}} \left( \sum_{i=1}^k B_i \right)^{\frac{1}{2}} \right\|.
$$
This is stronger than the inequality
$$
\left\| \sum_{i=1}^k A_i B_i \right\| \leq \left\| \left( \sum_{i=1}^k A_i \right) \left( \sum_{i=1}^k B_i \right) \right\|,
$$
which has been recently proved by Audenaert. Simple applications of these norm inequalities answer some questions of Bourin affirmatively.

Keywords and phrases: Unitarily invariant norm, Hilbert-Schmidt norm, singular value, majorization, positive semidefinite matrix, inequality.

REFERENCES


