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# WEIGHTED ITERATED HARDY–TYPE INEQUALITIES

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*Abstract.* In this paper reduction and equivalence theorems for the boundedness of the composition of a quasilinear operator $T$ with the Hardy and Copson operators in weighted Lebesgue spaces are proved. New equivalence theorems are obtained for the operator $T$ to be bounded in weighted Lebesgue spaces restricted to the cones of monotone functions, which allow to change the cone of non-decreasing functions to the cone of non-increasing functions and vice versa not changing the operator $T$. New characterizations of the weighted Hardy-type inequalities on the cones of monotone functions are given. The validity of so-called weighted iterated Hardy-type inequalities are characterized.

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