A MONOTONICITY PROPERTY INVOLVING THE
GENERALIZED ELLIPTIC INTEGRAL OF THE FIRST KIND

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Abstract. In this paper, we prove that the function

$$r \to Y(r) = \frac{\mathscr{K}_a(r)}{\sin(\pi a) r'^2 \log(e^{R(a)/2}/r')} - \frac{1}{r'^2}$$

is strictly increasing from $(0,1)$ onto $(\pi/[R(a)\sin(\pi a)] - 1, a(1-a))$ for all $a \in (0, 1/2]$, where $r' = \sqrt{1-r^2}$, $\mathscr{K}_a(r)$ is the generalized elliptic integral of the first kind, $R(a) = -2\gamma - \psi(a) - \psi(1-a)$, ψ is the classical psi function and $\gamma = 0.57721566\cdots$ is the Euler-Mascheroni constant.

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