Mathematical
Inequalities
& Applications
Volume 20, Number 3 (2017), 845–854
doi:10.7153/mia-20-53

SINGULAR MOSER–TRUDINGER INEQUALITY WITH THE EXACT GROWTH CONDITION IN $\mathbb{R}^n$

ZHAO LIU AND LU CHEN

Abstract. In this paper, we establish the singular Moser-Trudinger inequality with exact growth condition. We prove that there exists a positive constant C_n such that

$$\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t(1+|u|)^{\frac{n}{n-1}}}dx \leqslant C_n \int_{\mathbb{R}^n} \frac{|u(x)|^n}{|x|^t}dx$$

for all $u \in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |\nabla u|^n dx \leqslant 1$, where $\Phi(t) := e^t - \sum_{i=0}^{n-2} \frac{t^i}{i!}$. In order to avoid using symmetry and rearrangement, we employ the change of variables developed by Dong and Lu in [14] (see also Lam and Lu in [16]) to transform the singular Moser-Trudinger inequality with the exact growth condition to the corresponding non-singular case.

Mathematics subject classification (2010): 35A23, 42B37.
Keywords and phrases: Change of variables, Sharp constants, singular Moser-Trudinger inequality with the exact growth condition, Sobolev space.

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