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ITERATED HARDY–TYPE INEQUALITIES INVOLVING SUPREMA

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Abstract. In this paper, the boundedness of the composition of the supremal operators defined, for a non-negative measurable functions f on $(0,\infty)$, by

$$S_u g(t) := \operatorname*{ess\,sup}_{0<\tau\leqslant t} u(\tau) g(\tau),\ t \in (0,\infty),$$

and

$$S_u^* g(t) := \operatorname*{ess\,sup}_{t\leqslant\tau<\infty} u(\tau) g(\tau),\ t \in (0,\infty),$$

where u is a fixed continous weight on $(0,\infty)$, with the Hardy and Copson operators between weighted Lebesgue spaces $L^p(v)$ and $L^q(w)$ are characterized.

The complete solution of the restricted inequalities, that is, inequalities

$$\|S_u(f)\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)},$$

and

$$\|S_u(f)\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0.\infty)},$$

being satisfied on the cones of monotone functions f on $(0,\infty)$, are given.

Moreover, the complete characterization of the inequality

$$\|T_{u,b} f\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)},$$

being satisfied for every non-negative and non-increasing functions f on $(0,\infty)$, is given for $0 < p, q < \infty$, as well. Here the operator $T_{u,b}$ is defined for a measurable non-negative function f on $(0,\infty)$ by

$$(T_{u,b} g)(t) := \sup_{t\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(s) b(s)\, ds, \qquad t \in (0,\infty),$$

where u, b are two weight functions on $(0,\infty)$ such that u is continuous on $(0,\infty)$ and the function $B(t) := \int_0^t b(s)\, ds$ satisfies $0 < B(t) < \infty$ for every $t \in (0,\infty)$.

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