Mathematical
Inequalities
& Applications
Volume 20, Number 4 (2017), 1107–1120
doi:10.7153/mia-2017-20-71

SHARP GAUTSCHI INEQUALITY FOR PARAMETER $0 < p < 1$ WITH APPLICATIONS

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Abstract. In the article, we present the best possible parameters a,b on the interval $(0,\infty)$ such that the Gautschi double inequality $[(x^p+a)^{1/p}-x]/a < e^{x^p}\int_x^\infty e^{-t^p}dt < [(x^p+b)^{1/p}-x]/b$ holds for all $x>0$ and $p\in(0,1)$. As applications, we find new bounds for the incomplete gamma function $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$.

Mathematics subject classification (2010): 33B20, 26D07, 26D15.
Keywords and phrases: Incomplete gamma function, gamma function, psi function.

REFERENCES

[1] H. ALZER, *On some inequalities for the incomplete gamma function*, Math. Comp. **66**, (218) (1997), 771–778.
[2] H. ALZER AND Á. BARICZ, *Functional inequalities for the incomplete gamma function*, J. Math. Anal. Appl. **385**, (1) (2013), 167–178.
[3] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, (1997).
[4] J. M. BORWEIN AND O. Y. CHAN, *Uniform bounds for the complementary incomplete gamma function*, Math. Inequal. Appl. **12**, (1) (2009), 115–121.
[5] Y.-M. CHU, X.-M. ZHANG AND Z.-H. ZHANG, *The geometric convexity of a function involving gamma function with applications*, Commun. Korean Math. Soc. **25**, (3) (2010), 373–383.
[6] Á. ELBERT AND A. LAFORGIA, *An inequality for the product of two integrals related to the incomplete gamma function*, J. Inequal. Appl. **5**, (1) (2000), 39–51.
[7] G. FRANKLIN MONTGOMERY, *On the transmission error function for meteor-burst communication*, Proc. IRE. **46**, (1958), 1423–1424.
[8] W. GAUTSCHI, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. Phys. **38**, (1959/60), 77–81.
[9] B. GUHA, *Expansion of incomplete gamma function in terms of Jacobi polynomials*, Indian J. Pure Appl. Math. **12**, (8) (1981), 990–993.
[10] H.-L. GUPTA, *An integral equation involving incomplete gamma function as the kernel*, Indian J. Pure Appl. Math. **7**, (12) (1976), 1361–1366.
[11] S. S. GUPTA AND M. N. WAKNIS, *A system of inequalities for the incomplete gamma function and the normal integral*, Ann. Math. Statist. **36**, (1965), 139–149.
[12] A. GUTHMANN, *Asymptotische Entwicklungen für unvollständige Gammafunktionen*, Fourm Math. **3**, (2) (1991), 105–141.
[13] M. E. H. ISMALL AND A. LAFORGIA, *Functional inequalities for incomplete gamma and related functions*, Math. Inequal. Appl. **9**, (2) (2006), 299–302.
[14] R. N. KALIA AND S. KEITH, *Fractional calculus and expansions of incomplete gamma functions*, Appl. Math. Lett. **3**, (1) (1990), 19–21.
[15] S. H. KHAMIS, *Incomplete gamma functions expansions of statistical distribution functions*, Bull. Inst. Internat. Statist. **37**, (3) (1960), 385–396.
[16] W. F. KIBBLE, *A Bessel function in terms of incomplete Gamma functions*, J. Indian Math. Soc. **3**, (1) (1939), 271–294.

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[17] Y. KOMATU, *Elementary inequalities for Mills' ratio*, Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4**, (1955), 69–70.
[18] A. LAFORGIA AND P. NATALINI, *Supplements to known monotonicity results and inequalities for the gamma and incomplete gamma function*, J. Inequal. Appl. **2006**, Article ID 48727 (2006), 8 pages.
[19] Y. L. LUKE, *On the error in the Padé approximants for a form of the incomplete gamma function including the exponential function*, SIAM J. Math. Anal. **6**, (5) (1975), 829–839.
[20] B. A. MAMEDOV, *Analytical evaluation of the thermionic emission probability for semiconductor-insulator interfaces using incomplete gammam functions*, Appl. Math. Comput. **217**, (13) (2011), 6282–6285.
[21] M. R. MURTY AND E. SAHA, *Transcendental values of the incomplete gamma function and related questions*, Arch. Math. **105**, (3) (2015), 271–283.
[22] P. NATALINI AND B. PALUMBO, *Inequalities for the incomplete gamma function*, Math. Inequal. Appl. **3**, (1) (2000), 69–77.
[23] E. NEUMAN, *Inequalities and bounds for the incomplete gamma function*, Results Math. **63**, (3–4) (2013), 1209–1214.
[24] R. B. PARIS, *Error bounds for the uniform asymptotic expansion of the incomplete gamma function*, J. Comput. Appl. Math. **147**, (1) (2002), 215–231.
[25] R. B. PARIS, *A uniform asymptotic expansion for the incomplete gamma function*, J. Comput. Appl. Math. **148**, (2) (2002), 323–339.
[26] H. O. POLLAK, *A remark on "Elementary inequalities for Mills' ratio" by Yûsaku Komatu*, Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4**, (1956), 110–110.
[27] F. QI, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5**, (1) (2002), 61–67.
[28] F. QI AND S.-L. GUO, *Inequalities for the incomplete gamma and related functions*, Math. Inequal. Appl. **2**, (1) (1999), 47–53.
[29] F. QI AND J.-Q. MEI, *Some inequalities of the incomplete gamma and related functions*, Z. Anal. Anwendungen **18**, (3) (1999), 793–799.
[30] W. O. SCHUMANN, *Elektrische Durchbruchfeldstäke von Gasen*, Springer, Berlin, 1923.
[31] N. M. TEMME, *The asymptotic expansion of the incomplete gamma functions*, SIAM J. Math. Anal. **10**, (4) (1979), 757–766.
[32] M.-K. WANG AND Y.-M. CHU, *Refinements of transformation inequalities for zero-balanced hypergeometric functions*, Acta Math. Sci. **37B**, (3) (2017), 607–622.
[33] M.-K. WANG, Y.-M. CHU AND S.-L. QIU, *Sharp bounds for generalized elliptic integrals of the first kind*, J. Math. Anal. Appl. **429**, (2) (2015), 744–757.
[34] M.-K. WANG, Y.-M. CHU AND Y.-Q. SONG, *Asymptotical formulas for Gaussian and generalized hypergeometric functions*, Appl. Math. Comput. **276**, (2016), 44–60.
[35] M.-K. WANG, Y.-M. LI AND Y.-M. CHU, *Inequalities and infinite product formula for Ramanujan generalized modular equation function*, Ramanujan J. DOI 10.1007/s11139-017-9888-3.
[36] K. YAMAGATA, *A contribution to the theory of non-isothermal laminar flow of fluids inside a straight tube of circular cross section*, Mem. Fac. Engineering, Kyushu Imp. Univ. **8**, (1940), 365–449.
[37] Z.-H. YANG, *A new way to prove L'Hospital Monotone Rules with applications*, arXiv:1409.6408 [math.CA], available online at `http://arxiv.org/pdf/1409.6408v1.pdf`.
[38] Z.-H. YANG AND Y.-M. CHU, *Asymptotic forumlas for gamma function with applications*, Appl. Math. Comput. **270**, (2015), 665–680.
[39] Z.-H. YANG, Y.-M. CHU AND M.-K. WANG, *Monotonicity criterion for the quotient of power series with applications*, J. Math. Anal. Appl. **428**, (1) (2015), 587–604.
[40] Z.-H. YANG, W. ZHANG AND Y.-M. CHU, *Monotonicity and inequalities involving the incomplete gamma function*, J. Inequal. Appl. **2016**, Article 221 (2016), 10 pages.
[41] Z.-H. YANG, W. ZHANG AND Y.-M. CHU, *Monotonicity of the incomplete gamma function with applications*, J. Inequal. Appl. **2016**, Article 251 (2016), 10 pages.
[42] X.-M. ZHANG AND Y.-M. CHU, *A double inequality for gamma function*, J. Inequal. Appl. **2009**, Article ID 503782 (2009), 7 pages.
[43] T.-H. ZHAO AND Y.-M. CHU, *A class of logarithmicall complete monotonic functions associated with a gamma function*, J. Inequal. Appl. **2010**, Article ID 392431 (2010), 11 pages.
[44] T.-H. ZHAO, Y.-M. CHU AND Y.-P. JIANG, *Monotonic and logarithmically convex properties of a function involving gamma functions*, J. Inequal. Appl. **2009**, Article ID 728612 (2009), 13 pages.

[45] T.-H. ZHAO, Y.-M. CHU AND H. WANG, *Logarithmically complete monotonicity properties relating to the gamma function*, Abstr. Appl. Anal. **2011**, Article ID 896483 (2011), 13 pages.

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