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CONTINUITY AND APPROXIMATE DIFFERENTIABILITY OF MULTISUBLINEAR FRACTIONAL MAXIMAL FUNCTIONS

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Abstract. In this note we investigate the continuity and approximate differentiability of the m-sublinear fractional maximal operator

$$\mathfrak{M}_\alpha(\vec{f})(x)=\sup_{r>0}|B(x,r)|^{\alpha/d-m}\prod_{i=1}^{m}\int_{B(x,r)}|f_i(y)|dy,$$

where $m\geqslant 1$, $0\leqslant\alpha<md$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^1_{\rm loc}(\mathbb{R}^d)$. More precisely, we prove that $\mathfrak{M}_\alpha$ maps $W^{1,p_1}(\mathbb{R}^d)\times\cdots\times W^{1,p_m}(\mathbb{R}^d)$ into $W^{1,q}(\mathbb{R}^d)$ continuously, provided that $1<p_1,\dots,p_m<\infty$ and $0<\sum_{i=1}^m 1/p_i-\alpha/d=1/q\leqslant 1$. We also show that the multisublinear fractional maximal functions $\mathfrak{M}_\alpha(\vec{f})$ are approximately differentiable a.e. if $\vec{f}=(f_1,f_2,\dots,f_m)$ with each $f_j\in L^1(\mathbb{R}^d)$ being approximately differentiable a.e. As applications, the corresponding results for fractional maximal operators are established.

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REFERENCES

[1] D. R. ADAMS, *A note on Riesz potentials*, Duke Math. J., **42**, (1975), 765–778.
[2] D. R. ADAMS AND L. I. HEDBERG, *Function Spaces and Potential Theory*, Springer-Verlag, Berlin Heidelberg, 1996.
[3] J. M. ALDAZ, L. COLZANI AND J. PÉREZ LÁZARO, *Optimal bounds on the modulus of continuity of the uncentered Hardy-Littlewood maximal function*, J. Geom. Anal., **22**, 1 (2012), 132–167.
[4] J. M. ALDAZ AND J. PÉREZ LÁZARO, *Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities*, Trans. Amer. Math. Soc., **359**, 5 (2007), 2443–2461.
[5] J. M. ALDAZ AND J. PÉREZ LÁZARO, *Boundedness and unboundedness results for some maximal operators on functions of bounded variation*, J. Math. Anal. Appl., **337**, 1 (2008), 130–143.
[6] J. M. ALDAZ AND J. PÉREZ LÁZARO, *Regularity of the Hardy-Littlewood maximal operator on block decreasing functions*, Studia Math., **194**, (2009), 253–277.
[7] F. J. ALMGREN AND E. H. LIEB, *Symmetric decreasing rearrangement is sometimes continuous*, J. Amer. Math. Soc., **2**, (1989), 683–773.
[8] M. CAO AND Q. XUE, *Characterization of two-weighted inequalities for multilinear fractional maxiaml operator*, Non. Anal. TMA, **130**, (2016), 214–228.
[9] M. CAO AND Q. XUE, *Two-weight entropy boundedness of multilinear fractional type operators*, J. Math. Inequa., **11**, 2 (2017), 441–453.
[10] E. CARNEIRO AND J. MADRID, *Derivative bounds for fractional maximal functions*, Trans. Amer. Math. Soc., to appear.
[11] E. CARNEIRO AND D. MOREIRA, *On the regularity of maximal operators*, Proc. Amer. Math. Soc., **136**, 12 (2008), 4395–4404.
[12] E. CARNEIRO AND B.F. SVAITER, *On the variation of maximal operators of convolution type*, J. Funct. Anal., **265**, (2013), 837–865.

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[13] P. HAJŁASZ AND J. MALY, *On approximate differentiability of the maximal function*, Proc. Amer. Math. Soc., **138**, 1 (2010), 165–174.
[14] P. HAJŁASZ AND J. ONNINEN, *On boundedness of maximal functions in Sobolev spaces*, Ann. Acad. Sci. Fenn. Math., **29**, 1 (2004), 167–176.
[15] J. KINNUNEN, *The Hardy-Littlewood maximal function of a Sobolev function*, Israel J. Math., **100**, (1997), 117–124.
[16] J. KINNUNEN AND P. LINDQVIST, *The derivative of the maximal function*, J. Reine. Angew. Math., **503**, (1998), 161–167.
[17] J. KINNUNEN AND E. SAKSMAN, *Regularity of the fractional maximal function*, Bull. London Math. Soc., **35**, 4 (2003), 529–535.
[18] O. KURKA, *On the variation of the Hardy-Littlewood maximal function*, Ann. Acad. Sci. Fenn. Math., **40**, (2015), 109–133.
[19] A. K. LERNER, S. OMBROSI, C. PÉREZ, R. H. TORRES AND R. TRUJILLO-GONZÁLEZ, *New maximal functions and multiple weightes for the multilinear Calderón-Zygmund theory*, Adv. Math., **220**, (2009), 1222–1264.
[20] K. LI, K. MONE AND W. SUN, *Sharp weighted inequalities for multilinear fractional maximal operator and fractional integrals*, Math. Nachr., **288**, (5-6) (2015), 619–632.
[21] K. LI AND W. SUN, *Characterization of a two weight inequality for multilinear fractional maximal operators*, Houston J. Math., **42**, 3 (2016), 977–990.
[22] F. LIU, *A remark on the regularity of the discrete maximal operator*, Bull. Austral. Math. Soc., **95**, (2017), 108–120.
[23] F. LIU, T. CHEN AND H. WU, *A note on the endpoint regularity of the Hardy-Littlewood maximal functions*, Bull. Austral. Math. Soc., **94**, (2016), 121–130.
[24] F. LIU AND S. MAO, *On the regularity of the one-sided Hardy-Littlewood maximal functions*, Czech. Math. J., **67**, 142 (2017), 219–234.
[25] F. LIU AND H. WU, *On the regularity of the multisublinear maximal functions*, Canad. Math. Bull., **58**, 4 (2015), 808–817.
[26] F. LIU AND H. WU, *Endpoint regularity of multisublinear fractional maximal functions*, Canad. Math. Bull., to appear. DOI:10.4153/CMB-2016-044-9.
[27] F. LIU AND H. WU, *Regularity of discrete multisublinear fractional maximal functions*, Sci. China Math., to appear. DOI:10.1007/s11425-016-9011-2.
[28] F. LIU AND H. WU, *On the regularity of maximal operators supported by submanifolds*, J. Math. Anal. Appl., **453**, (2017), 144–158.
[29] H. LUIRO, *Continuity of the maixmal operator in Sobolev spaces*, Proc. Amer. Math. Soc., **135**, 1 (2007), 243–251.
[30] H. LUIRO, *On the regularity of the Hardy-Littlewood maximal operator on subdomains of* $\mathbb{R}^n$, Proc. Edinburgh Math. Soc., **53**, 1 (2010), 211–237.
[31] H. LUIRO AND J. NUUTINEN, *On the continuity of discrete maximal operators in Sobolev spaces*, Ann. Acad. Sci. Fenn. Math., **39**, (2014), 175–185.
[32] B. MUCKENHOUPT AND R. L. WHEEDEN, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192**, (1974), 261–274.
[33] C. PÉREZ AND R. WHEEDEN, *Potential operators, maximal functions, and generalizations of* A_∞, Poten. Anal., **19**, 1 (2003), 1–33.
[34] H. TANAKA, *A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function*, Bull. Austr. Math. Soc., **65**, 2 (2002), 253–258.
[35] H. WHITNEY, *On totally differentiable and smooth functions*, Pacific. J. Math., **1**, (1951), 143–159.