

## MONOTONICITY AND CONVEXITY OF THE RATIOS OF THE FIRST KIND MODIFIED BESSEL FUNCTIONS AND APPLICATIONS

ZHEN-HANG YANG AND SHEN-ZHOU ZHENG

*Abstract.* Let  $I_\nu(x)$  be modified Bessel functions of the first kind. We prove the monotonicity property of the function  $x \mapsto I_u(x)I_\nu(x)/I_{(u+\nu)/2}(x)^2$  on  $(0, \infty)$ . As a direct consequence, it deduces some known results including Turán-type inequalities and log-convexity or log-concavity of  $I_\nu$  in  $\nu$ , as well as it yields some new and interesting monotonicity and convexity concerning the ratios of modified Bessel functions of the first kind. In addition, a few of sharp bounds involving  $I_\nu(x)$  and their ratios are presented.

*Mathematics subject classification (2010):* 26A48, 26A51, 33C10, 33B10, 39B62.

*Keywords and phrases:* Modified Bessel functions of the first kind, monotonicity, convexity, functional inequality, Turán type inequality.

### REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 10th printing, Dover Publications, New York and Washington, 1972.
- [2] M. D. ALEXANDROV, A. A. LACIS, *A new three-parameter cloud/aerosol particle size distribution based on the generalized inverse Gaussian density function*, Appl. Math. Comput. **116** (2000), 153–165.
- [3] H. ALZER AND G. FELDER, *A Turán-type inequality for the gamma function*, J. Math. Anal. Appl. **350** (2009), 276–282.
- [4] D. E. AMOS, *Computation of modified Bessel functions and their ratios*, Math. Comp. **28** (1974), 239–251.
- [5] Á. BARICZ, E. NEUMAN, *Inequalities involving modified Bessel functions of the first kind II*, J. Math. Anal. Appl. **332** (2007), 265–271.
- [6] Á. BARICZ, *Turán type inequalities for hypergeometric functions*, Proc. Amer. Math. Soc. **136** (2008), 3223–3229.
- [7] Á. BARICZ, *Functional inequalities involving Bessel and modified Bessel functions of the first kind*, Expo. Math. **26** (2008), 279–293.
- [8] Á. BARICZ, *On a product of modified Bessel functions*, Proc. Amer. Math. Soc. **137** (2009), 189–193.
- [9] Á. BARICZ, *Turán type inequalities for some probability density functions*, Studia Sci. Math. Hungar. **47** (2010), 175–189.
- [10] Á. BARICZ, *Turán type inequalities for modified Bessel functions*, Bull. Aust. Math. Soc. **82** (2010), 254–264.
- [11] Á. BARICZ, *Bounds for modified Bessel functions of the first and second kinds*, Proc. Edinburgh Math. Soc. **53** (2010), 575–599.
- [12] Á. BARICZ AND S. PONNUSAMY, *On Turán type inequalities for modified Bessel functions*, Proc. Amer. Math. Soc. **141** (2013), 523–532.
- [13] Á. BARICZ AND T. K. POGÁNY, *Turán determinants of Bessel functions*, Forum Math. **26** (2014), 295–322.
- [14] Á. BARICZ, *Bounds for Turánians of modified Bessel functions*, Expo. Math. **2015** (2015), 223–251.
- [15] M. BIERNACKI AND J. KRZYZ, *On the monotonicity of certain functionals in the theory of analytic functions*, Annales Universitatis Mariae Curie-Skłodowska **9** (1955), 135–147.

- [16] L. DEVROYE, *Simulating Bessel random variables*, Statist. Probab. Lett. **57** (2002), 249–257.
- [17] T. H. GRONWALL, *An inequality for the Bessel functions of the first kind with imaginary argument*, Ann. Math. **33** (1932), 275–278.
- [18] K. HORNIK AND B. GRÜN, *Amos-type bounds for modified Bessel function ratios*, J. Math. Anal. Appl. **408** (2013), 91–101.
- [19] M. E. H. ISMAIL, *Bessel functions and the infinite divisibility of the Student  $t$ -distribution*, Ann. Probab. **5** (1977), 582–585.
- [20] M. E. H. ISMAIL, A. LAFORGIA, *Monotonicity properties of determinants of special functions*, Constr. Approx. **26** (2007), 1–9.
- [21] C. M. JOSHI AND S. K. BISSU, *Some inequalities of Bessel and modified Bessel functions*, J. Aust. Math. Soc. Ser. A **50** (1991), 333–342.
- [22] S. I. KALMYKOV, D. B. KARP, *Log-concavity for series in reciprocal gamma functions and applications*, Integral Transforms Spec. Funct. **24** (2013), 859–872.
- [23] D. K. KAZARINOFF, *On Wallis formula*, Edinburgh. Math. Soc. Notes **40**(1956), 19–21.
- [24] P. A. KHAZRON, I. W. SELESNICK, *Bayesian estimation of Bessel  $K$  form random vectors in AWGN*, IEEE Signal Process. Lett. **15** (2008), 261–264.
- [25] C. G. KOKOLOGIANNAKI, *Bounds for functions involving ratios of modified Bessel functions*, J. Math. Anal. Appl. **385** (2012), 737–742.
- [26] A. LAFORGIA, P. NATALINI, *Turán-type inequalities for some special functions*, JIPAM. J. Inequal. Pure Appl. Math. **7** (2006), Art. 22, 3 pp.
- [27] L. LORCH, *Monotonicity of the zeros of a cross product of Bessel functions*, Methods Appl. Anal. **1** (1994), 75–80.
- [28] A. A. LUSHNIKOV, J. S. BHATT, I. J. FORD, *Stochastic approach to chemical kinetics in ultrafine aerosols*, J. Aerosol Sci. **34** (2003), 1117–1133.
- [29] I. MEZŐ, Á. BARICZ, *Properties of the Turánian of modified Bessel functions*, Math. Inequal. Appl. **20** (2017), 991–1001.
- [30] I. NÄSELL, *Schistosomiasis in a community with external infection*, Proc. 8th Internat. Biometric Conf., Editura Academiei Republicii Socialiste Romania (1975), 123–131.
- [31] I. NÄSELL AND W. M. HIRSCH, *The transmission dynamics of schistosomiasis*, Comm. Pure Appl. Math. **26** (1973), 395–453.
- [32] E. NEUMAN, *Inequalities involving modified Bessel functions of the first kind*, J. Math. Anal. Appl. **171** (1992), 532–536.
- [33] R. S. PHILLIPS AND H. MALIN, *Bessel function approximations*, Amer. J. Math. **72** (1950), 407–418.
- [34] C. ROBERT, *Modified Bessel functions and their applications in probability and statistics*, Statist. Probab. Lett. **9** (1990), 155–161.
- [35] J. SEGURA, *Bounds for ratios of modified Bessel functions and associated Turán-type inequalities*, J. Math. Anal. Appl. **374** (2011), 516–528.
- [36] H. C. SIMPSON, S. J. SPECTOR, *Some monotonicity results for ratios of modified Bessel functions*, Quart. Appl. Math. **42** (1984), 95–98.
- [37] H. C. SIMPSON, S. J. SPECTOR, *On barreling for a special material in finite elasticity*, Quart. Appl. Math. **42** (1984), 99–105.
- [38] S. TAN, L. JIAO, *Multishrinkage: Analytical form for a Bayesian wavelet estimator based on the multivariate Laplacian model*, Optim. Lett. **32** (2007), 2583–2585.
- [39] V. R. THIRUVENKATACHAR AND T. S. NANJUNDIAH, *Inequalities concerning Bessel functions and orthogonal polynomials*, Proc. Ind. Acad. Sci. Sect. A **33** (1951), 373–384.
- [40] H. VAN HAERINGEN, *Bound states for  $r - 2$ -like potentials in one and three dimensions*, J. Math. Phys. **19** (1978), 2171–2179.
- [41] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1922.
- [42] ZH.-H. YANG AND Y.-M. CHU, *On approximating the modified Bessel function of the first kind and Toader- $Q_j$  mean*, J. Inequal. Appl. **2016** (2016): 40, DOI 10.1186/s13660-016-0988-1.
- [43] ZH.-H. YANG AND SH.-ZH. ZHENG, *The monotonicity and convexity for the ratios of modified Bessel functions of the second kind and applications*, Proc. Amer. Math. Soc. **145** (2017), 2943–2958.