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LYAPUNOV–TYPE INEQUALITY FOR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH HILFER DERIVATIVE

NIMISHA PATHAK

Abstract. In this work, we establish Lyapunov-type inequalities for fractional boundary value problems containing Hilfer derivative of order α, $1 < \alpha \leqslant 2$ and type $0 \leqslant \beta \leqslant 1$. We consider the boundary value problems with the Dirichlet, and a mixed set of Dirichlet and Neumann boundary conditions. We consider both integer and fractional order eigenvalue problems, determine a lower bound for the smallest eigenvalue using Lyapunov-type inequalities, and improve these bounds using semi maximum norm and Cauchy-Schwarz inequalities. We use the improved lower bounds to obtain intervals where a certain Mittag-Leffler functions have no real zeros. Further, we discuss the particular cases for the type $\beta = 0$ and $\beta = 1$, which give the results respectively for Riemann-Liouville and Caputo fractional boundary value as well as eigenvalue problems. For both the fractional and the integer order eigenvalue problems, we give a comparison between the smallest eigenvalue and its lower bounds obtained from the Lyapunov-type, semi maximum norm and Cauchy-Schwarz inequalities. Results show that the Lyapunov-type inequality gives the worse and semi maximum norm and Cauchy-Schwarz inequalities give better lower bound estimates for the smallest eigenvalues.

Mathematics subject classification (2010): 34A08, 34A40, 26D10, 33E12.
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