

## LYAPUNOV-TYPE INEQUALITY FOR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH HILFER DERIVATIVE

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*Abstract.* In this work, we establish Lyapunov-type inequalities for fractional boundary value problems containing Hilfer derivative of order  $\alpha$ ,  $1 < \alpha \leq 2$  and type  $0 \leq \beta \leq 1$ . We consider the boundary value problems with the Dirichlet, and a mixed set of Dirichlet and Neumann boundary conditions. We consider both integer and fractional order eigenvalue problems, determine a lower bound for the smallest eigenvalue using Lyapunov-type inequalities, and improve these bounds using semi maximum norm and Cauchy-Schwarz inequalities. We use the improved lower bounds to obtain intervals where a certain Mittag-Leffler functions have no real zeros. Further, we discuss the particular cases for the type  $\beta = 0$  and  $\beta = 1$ , which give the results respectively for Riemann-Liouville and Caputo fractional boundary value as well as eigenvalue problems. For both the fractional and the integer order eigenvalue problems, we give a comparison between the smallest eigenvalue and its lower bounds obtained from the Lyapunov-type, semi maximum norm and Cauchy-Schwarz inequalities. Results show that the Lyapunov-type inequality gives the worse and semi maximum norm and Cauchy-Schwarz inequalities give better lower bound estimates for the smallest eigenvalues.

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