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LÉVY–KHINTCHINE REPRESENTATION OF TOADER–QI MEAN

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Abstract. In the paper, by virtue of a Lévy-Khintchine representation and an alternative integral representation for the weighted geometric mean, the authors establish a Lévy-Khintchine representation and an alternative integral representation for the Toader-Qi mean, verify that the Toader-Qi mean is a Bernstein function and that the divided difference of the Toader-Qi mean is a Stieltjes function, and collect a probabilistic interpretation and an application in engineering of the Toader-Qi mean.

Mathematics subject classification (2010): Primary 44A15, Secondary 26E60, 30E20, 33C10, 60G50.
Keywords and phrases: Lévy-Khintchine representation, integral representation, Bernstein function, Stieltjes function, Toader-Qi mean, weighted geometric mean, Bessel function of the first kind, probabilistic interpretation, application in engineering, inequality.

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