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ON APPROXIMATING THE ERROR FUNCTION

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Abstract. In the article, we find the best possible parameters p and q on the interval $(7/5,(7\pi-20)/(5\pi-15))$ such that the double inequality

$$\sqrt{1-\lambda(p)e^{-px^2}-(1-\lambda(p))e^{-\mu(p)x^2}}<\operatorname{erf}(x)<\sqrt{1-\lambda(q)e^{-qx^2}-(1-\lambda(q))e^{-\mu(q)x^2}}$$

holds for all $x>0$, where $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the error function, $\lambda(p)=4[(7\pi-20)-5(\pi-3)p]/[\pi(15p^2-40p+28)]$, $\mu(p)=4(5p-7)/[5(3p-4)]$.

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