

LANDEN INEQUALITIES FOR A CLASS OF HYPERGEOMETRIC FUNCTIONS WITH APPLICATIONS

MIAO-KUN WANG AND YU-MING CHU

Abstract. In this paper, we study a class of Gaussian hypergeometric function ${}_2F_1(a, b; (a+b+1)/2; x)$ ($a, b > 0$), and find the maximal regions of ab plane in the first quadrant where the well-known Landen identities for the complete elliptic integrals of the first kind turn on respective inequalities valid for each $x \in (0, 1)$. Besides, the generalized Grötzsch ring function with two parameters $\mu_{a,b}(r)$ is introduced, and the analogs of duplication formula satisfied by Grötzsch ring function $\mu(r)$ for $\mu_{a,b}(r)$, in the form of inequalities, will be derived.

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