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ON THE BEST HARDY CONSTANT FOR QUASI–ARITHMETIC MEANS AND HOMOGENEOUS DEVIATION MEANS

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Abstract. The aim of this paper is to characterize the so-called Hardy means, i.e., those means $M\colon \bigcup_{n=1}^{\infty}\mathbb{R}_+^n \to \mathbb{R}_+$ that satisfy the inequality

$$\sum_{n=1}^{\infty} M(x_1,\dots,x_n) \leqslant C\sum_{n=1}^{\infty} x_n$$

for all positive sequences (x_n) with some finite positive constant C. The smallest constant C satisfying this property is called the Hardy constant of the mean M.

In this paper we determine the Hardy constant in the cases when the mean M is either a concave quasi-arithmetic or a concave and homogeneous deviation mean.

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