

INFINITE SERIES FORMULA FOR HÜBNER UPPER BOUND FUNCTION WITH APPLICATIONS TO HERSCH–PFLUGER DISTORTION FUNCTION

MIAO-KUN WANG, SONG-LIANG QIU AND YU-MING CHU

Abstract. In this paper, we find an infinite series formula for the Hübner upper bound function, present the bounds for the Hersch-Pfluger distortion function, and improve the well known results on the quasiconformal Schwarz lemma and the solutions of the Ramanujan modular equations. Besides, the submultiplicative and power submultiplicative properties of the Hersch-Pfluger distortion function are also discussed.

Mathematics subject classification (2010): 33C62, 33E05.

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