

IMPROVED HARDY AND RELICH TYPE INEQUALITIES WITH TWO WEIGHT FUNCTIONS

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Abstract. In this work, we obtain several improved versions of two weight Hardy and Rellich type inequalities on the sub-Riemannian manifold \mathbb{R}^{2n+1} defined by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2} \frac{\partial}{\partial l}, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2} \frac{\partial}{\partial l}, \quad j = 1, 2, \dots, n$$

where $(z, l) = (x, y, l) \in \mathbb{R}^{2n+1}$, $|z| = (|x|^2 + |y|^2)^{1/2}$ and $k \geq 1$.

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