

MONOTONICITY PROPERTIES AND BOUNDS INVOLVING THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

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Abstract. In the article, we establish several monotonicity properties of the functions involving the complete elliptic integral of the first kind. As applications, we present sharp bounds for the complete elliptic integral of the first kind and the arithmetic-geometric mean.

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