COMPLETE MONOTONICITY AND INEQUALITES
INVOLVING GURLAND'S RATIOS OF GAMMA FUNCTIONS

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Abstract. In this paper, by a comparison inequality for an auxiliary function with two parameters, we present necessary and sufficient conditions for four classes of ratios involving gamma function to be logarithmically completely monotonic. These not only greatly generalize and improve certain known results, but also yield many new inequalities for gamma, psi and polygamma functions.

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