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CHARACTERIZATION OF OPERATOR CONVEX FUNCTIONS BY CERTAIN OPERATOR INEQUALITIES

HIROYUKI OSAKA, YUKIHIRO TSURUMI AND SHUHEI WADA

Abstract. For $\lambda \in (0,1)$, let ψ be a non-constant, non-negative, continuous function on $(0,\infty)$ and let $\Gamma_\lambda(\psi)$ be the set of all non-trivial operator means σ such that an inequality

$$\psi(A\nabla_\lambda B) \leqslant \psi(A)\sigma\psi(B)$$

holds for all $A,B \in B(H)^{++}$. Then we have:

1. ψ is a decreasing operator convex function if and only if

$$\Gamma_\lambda(\psi) = \{\sigma \mid !_\lambda \leqslant \sigma \leqslant \nabla_\lambda\}.$$

2. ψ is an operator convex function which is not a decreasing function if and only if

$$\Gamma_\lambda(\psi) = \{\nabla_\lambda\}.$$

The first result is a weighted version of Ando and Hiai's characterization of an operator monotone decreasing function and these two results imply each other.

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