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MONOTONICITY AND INEQUALITIES INVOLVING ZERO–BALANCED HYPERGEOMETRIC FUNCTION

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Abstract. In the article, we present a monotonicity property involving the zero-balanced hypergeometric function $F(a,b;a+b;x)$ for all $a,b>0$, and establish several sharp inequalities for $F(a,b;a+b;x)$ in the first quadrant of ab-plane, which are the generalizations of the previously results.

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