

FRACTIONAL ORDER HARDY–TYPE INEQUALITY IN FRACTIONAL h -DISCRETE CALCULUS

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Abstract. We investigate the power weights fractional order Hardy-type inequality in the following form:

$$\left(\int_0^{\infty} \int_0^{\infty} \frac{|f(x) - f(y)|^p}{|x - y|^{1+p\alpha}} dx dy \right)^p \leq C \left(\int_0^{\infty} |f'(x)|^p x^{(1-\alpha)p} dx \right)^p$$

for $0 < \alpha < 1$ and $1 < p < \infty$ in fractional h -discrete calculus, where $C = \frac{2^{\frac{1}{p}} \alpha^{-1}}{(p-\alpha)^{\frac{1}{p}}}$. For h -fractional function we prove a discrete analogue of above inequality in fractional h -discrete calculus, is proved and discussed. Moreover, we prove that the same constant is sharp also in this case.

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