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EXTREMAL FUNCTIONS FOR THE MODIFIED TRUDINGER–MOSER INEQUALITIES IN TWO DIMENSIONS

YAMIN WANG

Abstract. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $W_0^{1,2}(\Omega)$ be the standard Sobolev space. Assuming certain conditions on a function $g : \mathbb{R} \to \mathbb{R}$, we prove that the supremum

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|\nabla u\|_2 \leqslant 1} \int_\Omega (1+g(u))e^{4\pi u^2} dx,$$

can be attained by some function $u_0 \in W_0^{1,2}(\Omega)$ with $\|\nabla u_0\|_2 = 1$. The proof is based on the usual blow-up analysis. Also we consider the same problem for the supremum

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|\nabla u\|_2 \leqslant 1} \int_\Omega h(1+g(u))e^{4\pi u^2} dx,$$

where h is continuous in $\overline{\Omega}$, $h \geqslant 0$ and $h \not\equiv 0$.

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