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A HARMONIC MEAN INEQUALITY FOR THE POLYGAMMA FUNCTION

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Abstract. In this work, we discuss some new inequalities and a concavity property of the polygamma function $\psi^{(n)}(x) = \dfrac{d^n}{dx^n}\psi(x)$, $x > 0$, where $\psi(x)$ represents the digamma function (i.e. logarithmic derivative of the gamma function $\Gamma(x)$). Using these inequalities, minimum value of harmonic mean of $(-1)^n\psi^{(n)}(x)$ and $(-1)^n\psi^{(n)}(1/x)$ is obtained in terms of the Riemann zeta function and the Bernoulli numbers. Further new characterizations of π and the Apéry's constant are also presented as a consequence.

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