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NOTES ON THE COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

ZHEN-HANG YANG, WEI-MAO QIAN, WEN ZHANG AND YU-MING CHU

Abstract. In the article, we present several monotonicity properties and bounds for the complete elliptic integral of the first kind. As applications, we find sharp bounds for the arithmetic-geometric mean.

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