Mathematical
Inequalities
& Applications
Volume 23, Number 4 (2020), 1391–1423
doi:10.7153/mia-2020-23-101

EXTENSIONS OF QUADRATIC TRANSFORMATION IDENTITIES FOR HYPERGEOMETRIC FUNCTIONS

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Abstract. In the article, we extend the identities $F_0(x)=(1+r)F_0(r)$, $2F_0(\sqrt{1-x})=(1+r)F_0(1-r^2)$, $2\overline{F}_0(y)=\sqrt{1+3r}\,\overline{F}_0(1-r^2)$ and $\overline{F}_0(1-y)=\sqrt{1+3r}\,\overline{F}_0(r^2)$ for hypergeometric functions $F_0(r)={}_2F_1(1/2,1;3/2;r)$ and $\overline{F}_0(r)={}_2F_1(1/4,3/4;1;r)$, performed by the quadratic transformations $r\mapsto x=4r/(1+r)^2$, $r\mapsto\sqrt{1-x}$, $r\mapsto y=(1-r)^2/(1+3r)^2$ and $r\mapsto 1-y$, to the zero-balanced hypergeometric function ${}_2F_1(a,b;a+b;r)$, by showing new properties of ${}_2F_1(a,b;a+b;r)$ and the Ramanujan type constant, and the monotonicity properties of certain combinations in terms of hypergeometric and elementary functions. These extensions give complete solutions of the problem of extending the transformation identities abovementioned to ${}_2F_1(a,b;a+b;r)$, and perfect all the known related results. By these results, sharp transformation inequalities are obtained for the generalized Grötzsch ring function appearing in Ramanujan's modular equations.

Mathematics subject classification (2010): 33C05, 33F05, 30C62, 11F03.

Keywords and phrases: Zero-balanced hypergeometric function, quadratic transformation, generalized Grötzsch ring function, monotonicity, extension.

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Paper MIA-23-101

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