

SHARP SOBOLEV INEQUALITIES ON THE COMPLEX SPHERE

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Abstract. This paper is devoted to establish a class of sharp Sobolev inequalities on the unit complex sphere as follows:

1) **Case** $0 < d < Q = 2n + 2$: for any $f \in C^\infty$ and $2 \leq q \leq \frac{2Q}{Q-d}$,

$$\|f\|_q^2 \leq \frac{8(q-2)}{d(Q-d)} \frac{\Gamma^2((Q-d)/4+1)}{\Gamma^2((Q+d)/4)} \left(\int_{\mathbb{S}^{2n+1}} f \mathcal{A}_d f d\xi - \frac{\Gamma^2((Q+d)/4)}{\Gamma^2((Q-d)/4)} \int_{\mathbb{S}^{2n+1}} |f|^2 d\xi \right) + \int_{\mathbb{S}^{2n+1}} |f|^2 d\xi;$$

2) **Case** $d = Q$: for any $f \in C^\infty \cap \mathbb{R}\mathcal{P}$ and $2 \leq q < +\infty$,

$$\|f\|_q^2 \leq \frac{q-2}{(n+1)!} \int_{\mathbb{S}^{2n+1}} f \mathcal{A}'_Q f d\xi + \int_{\mathbb{S}^{2n+1}} |f|^2 d\xi,$$

where \mathcal{A}_d ($0 < d < Q$) are the intertwining operator, \mathcal{A}'_Q is the conditional intertwinor introduced in [2], and $d\xi$ is the normalized surface measure of \mathbb{S}^{2n+1} .

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