Mathematical
Inequalities
& Applications
Volume 23, Number 3 (2020), 843–853
doi:10.7153/mia-2020-23-68

SHARP RATIONAL BOUNDS FOR THE GAMMA FUNCTION

JIAN-MEI SHEN, ZHEN-HANG YANG, WEI-MAO QIAN, WEN ZHANG AND YU-MING CHU

Abstract. In the article, we prove that the inequality

$$\Gamma(x+1) \leqslant \frac{x^2+p}{x+p}$$

holds for all $x \in (0,1)$ if and only if $p \geqslant p_0$, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the gamma function, $p_0 = \left[x_0\Gamma(x_0+1) - x_0^2\right]/[1-\Gamma(x_0+1)] = 1.755\cdots$, $x_0 = 0.192\cdots$ is the unique solution of the equation $\psi(x+1) = [1-\Gamma(x+1)][2-\Gamma(x)]/[(1-x)\Gamma(x+1)]$ on the interval $(0,1)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the psi function. As applications, we present the best possible parameters λ_0 and μ_0 on the interval $(0,\infty)$ such that the double inequality

$$\frac{x^2+\lambda_0}{x+\lambda_0} < \Gamma(x+1) < \frac{x^2+\mu_0}{x+\mu_0}$$

holds for all $x \in (1/2,1)$, and the two-sided inequality

$$\frac{\pi x(1-x)(1-x+\mu_0)}{\sin(\pi x)[(1-x)^2+\mu_0]} < \Gamma(x+1) < \frac{\pi x(1-x)(1-x+\lambda_0)}{\sin(\pi x)[(1-x)^2+\lambda_0]}$$

takes place for all $x \in (0,1/2)$.

Mathematics subject classification (2010): 41A60, 33B15, 26D07.
Keywords and phrases: Gamma function, psi function, rational bound, Euler-Mascheroni constant.

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