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CHARACTERIZATIONS OF WEIGHTED HARDY–RELLICH INEQUALITIES AND THEIR APPLICATIONS

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Abstract. Let Ω be a bounded open domain in $\mathbb{R}^n$. We establish characterizations of the weighted Hardy-Rellich inequalities that connect the integrals over Ω of the first and second derivatives of the considered functions, via some weighted vector-valued Hardy inequalities and weighted dual Hardy inequalities. These characterizations are then applied to derive some new weighted Rellich inequalities with homogenous weights that admit singularities on unit sphere $\mathbb{S}^{n-1}$.

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