IMPROVED HARDY INEQUALITIES WITH EXACT REMAINDER TERMS

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Abstract. We set up several identities that imply some versions of the Hardy type inequalities. These equalities give a straightforward understanding of several Hardy type inequalities as well as the nonexistence of nontrivial optimizers. These identities also provide the "virtual" extremizers for many Hardy type inequalities.

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