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TWO–WEIGHTED INEQUALITIES FOR THE FRACTIONAL INTEGRAL ASSOCIATED TO THE SCHRÖDINGER OPERATOR

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Abstract. In this article we prove that the fractional integral operator associated to the Schrödinger second order differential operator $\mathscr{L}^{-\alpha/2} = (-\Delta + V)^{-\alpha/2}$ maps with continuity weak Lebesgue space $L^{p,\infty}(v)$ into weighted Campanato-Hölder type spaces $BMO^{\beta}_{\mathscr{L}}(w)$, thus improving regularity under appropriate conditions on the pair of weights (v,w) and the parameters p, α and β. We also prove the continuous mapping from $BMO^{\beta}_{\mathscr{L}}(v)$ to $BMO^{\gamma}_{\mathscr{L}}(w)$ for adequate pair of weights. Our results improve those known for the same weight in both sides of the inequality and they also enlarge the families of weights known for the classical fractional integral associated to the Laplacian operator $\mathscr{L} = -\Delta$.

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1. Introduction

Regularity estimates of solutions of second order differential operators are central in the study of partial differential equations. Sometimes these results are closely related to regularity estimates for negative powers of those operators. Keeping in mind this fact in this paper we focus our attention on estimates on Campanato-Hölder type spaces of fractional integrals -negative powers- of the Schrödinger differential operator

$$\mathscr{L} = -\Delta + V,$$

on $\mathbb{R}^d$ with $d \geqslant 3$, where the potential $V \geqslant 0$ belongs to a reverse Hölder class RH_q, for some exponent $q > \frac{d}{2}$, as defined in (1.4). For a deeper insight in this direction see [14, 24, 25].

At this point we must recall that the Hölder-α continuous space of functions f such that $\|f\|_{\mathscr{C}^\alpha} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$, $0 < \alpha \leqslant 1$, can be identified with the Campanato space BMO^α defined by the seminorm $\|f\|_{BMO^\alpha} = \sup_B \frac{1}{|B|^{1+\alpha/\lambda}} \int_B |f(x) - f_B|dx$, where the supremum is taken over all the balls $B \in \mathbb{R}^d$ and f_B is the mean value of f on B, see for example [38, 22, 26]. In the Schrödinger setting analogous result was obtained in [2] by identifying a Campanato-type space $BMO^{\alpha}_{\mathscr{L}}$ with certain type Hölder-α continuous space, see next section. This identification will be the key tool to interpret the information given by our results in terms of regularity.

Let us recall that negative powers of the Schrödinger operator can be expressed in terms of the heat diffusion semigroup generated by $\mathscr{L}$, $e^{-t\mathscr{L}}$, as

$$\mathscr{I}_\alpha f(x) = \mathscr{L}^{-\alpha/2} f(x) = \int_0^\infty e^{-t\mathscr{L}} f(x) t^{\alpha/2} \frac{dt}{t}, \quad \alpha > 0. \tag{1.1}$$

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The above operator is also named (Schrödinger) fractional integral operator of order α. When $V = 0$ then $\mathscr{L} = -\Delta$ is the Laplacian operator and we have the classical fractional integral I_α.

For each $t > 0$ the operator $e^{-t\mathscr{L}}$ is an integral operator with kernel $k_t(x,y)$ having a better behaviour far away from the diagonal $\{(x,x),\ x \in \mathbb{R}^d\}$ than the heat diffusion kernel $\frac{1}{(4\pi t)^{d/2}} e^{-|x-y|^2/4t}$ associated to $-\Delta$, see Lemma 2.1 below and [12], [13] and [23]. It follows from this property that $\mathscr{I}_\alpha f$ is finite a.e. for $f \in L^p$ with $p \geqslant 1$.

When applied on L^p-spaces the value $p = d/\alpha$ constitutes a breaking-point for the classical fractional integral I_α. More clearly, for $p < d/\alpha$ it satisfies a (p,q)-norm inequality with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, see [18, 19, 37, 21], but for $p \geqslant d/\alpha$ I_α instead shows regularity properties of the form

$$\frac{1}{|B|^{1+\frac{\alpha}{d}-\frac{1}{p}}} \int_B |I_\alpha f(x) - c_{B,f}|\,dx \leqslant C\|f\|_p \tag{1.2}$$

for $f \in L^p$, any ball B and some constant $c_{B,f}$. In other words $I_\alpha f$ belongs to the Campanato space BMO^β with $\beta = \frac{\alpha}{d} - \frac{1}{p}$, see for example [38, 22, 29]. In particular, when $p = d/\alpha$ the arrival space is $BMO^0 = BMO$, the well known space of bounded mean oscillation functions of John and Nirenberg.

The Schrödinger fractional integral $\mathscr{I}_\alpha$ behaves similarly to I_α when $p < d/\alpha$ in the sense that it also maps L^p in L^q with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ but for $p = d/\alpha$ the behavior of $\mathscr{I}_\alpha$ is better since it maps $L^{d/\alpha}$ into a space, denoted $BMO_{\mathscr{L}}$, which is in fact smaller than the classical BMO-space, see [10]. Finally for $p > d/\alpha$ the operator $\mathscr{I}_\alpha$ maps with continuity L^p into a Campanato-type space $BMO^\beta_{\mathscr{L}}$, see [2, 3, 24]. This space is shown to be the dual of the H^p-space introduced in [11] and [13], as it can be easily checked from the atomic decomposition given there. For definition and properties of weighted H^p-spaces see [1].

Going back again to the classical setting, Harboure, Viviani and Salinas in [17] obtained a more general estimate than (1.2) with I_α defined on a weak weighted L^p-space. That is

$$\frac{1}{v(B)|B|^{\frac{\alpha}{d}-\frac{1}{p}}} \int_B |I_\alpha f(x) - c_{B,f}|\,dx \leqslant C\left[\frac{f}{v}\right]_p \tag{1.3}$$

for an adequate class of weights $H(\alpha,p)$.

We recall that the weak weighted $L^{p,\infty}(v)$, $p > 1$, is the space of measurable functions f such that $[\frac{f}{v}]_p = (\sup_{t>0} t^p |\{x\colon \frac{|f(x)|}{v(x)} > t\}|)^{\frac{1}{p}} < \infty$ where v is a measurable non-negative function and that the expression on the left side of (1.3) represents a semi-norm in a weighted $BMO^{\alpha-d/p}(v)$, see also [35] and references therein.

The class of weights $H(\alpha,p)$ introduced in [17] was later used in [2] to derive for the Schrödinger fractional integral $\mathscr{I}_\alpha$ estimates, one of which is of the type of (1.3), that show the continuity of the operator from $L^{p,\infty}(v)$ into a weighted Campanato-type space $BMO^\beta_{\mathscr{L}}(v)$. However, since the kernel of $\mathscr{I}_\alpha$ behaves away from the diagonal better than the kernel of the classical fractional integral, it is natural to wonder if there exists a wider class of weights than those used in [2] from which the same continuity result can be deduced.

The first kind of estimates in the present paper gives a positive answer to this question. Moreover, it deals with the two-weight version of the boundedness results obtained in [2].

Our two-weighted results involve hypothesis based on a "power bump" property. This type of conditions already appeared in several papers dealing with two-weighted inequalities, see for instance [32, 15]. Even a weaker "log bump" condition also appeared, for example, in [7, 8, 27, 9] and references therein. For two-weighted inequalities for classical potential operators see for instance [5, 31, 30, 6, 34, 33, 28], and for the Schrödinger fractional integral and maximal operators associated see [20].

The purpose of those hypothesis is to get simpler conditions on the weights by avoiding extra assumptions or conditions involving the operators under consideration.

In order to introduce our main results we turn our attention to the Schrödinger operator $\mathscr{L} = -\Delta + V$. We say that the function V belongs to a reverse Hölder class of order q denoted by RH_q for some $q > \frac{d}{2}$ if

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{\frac{1}{q}} \leqslant \frac{C}{|B|}\int_B V(y)\,dy \tag{1.4}$$

for any ball $B \subset \mathbb{R}^d$. In the sequel we denote $q_0 = \sup\{q : V \in RH_q\}$ and $\delta_0 = \min(1, 2 - \frac{d}{q_0})$.

A "critical radii" function associated to V is defined by

$$\rho(x) = \sup\left\{r > 0 :\ \frac{1}{r^{d-2}}\int_{B(x,r)} V \leqslant 1\right\}, \quad x \in \mathbb{R}^d. \tag{1.5}$$

Such function is finite for all $x \in \mathbb{R}^d$ and plays an important role in the description of the spaces and, hence, in the inequalities related to regularity of the operators acting on or arriving at these spaces associated to $\mathscr{L}$, see [10, 12, 13, 36].

We also denote by L^1_{loc} the set of locally integrable functions of $\mathbb{R}^d$. By a weight we mean a locally integrable function $w > 0$ a.e. and along this work we denote $w(E) = \int_E w(x)dx$ for any measurable subset $E \in \mathbb{R}^d$.

Given $\eta \geqslant 1$, a weight w belongs to the class D_η, $w \in D_\eta$, if there exists a constant C such that $w(tB) \leqslant Ct^{d\eta}w(B)$ for any ball $B \subset \mathbb{R}^d$ and $t \geqslant 1$. It is easy to see that a weight w belongs to $D = \cup_{\eta \geqslant 1} D_\eta$ if and only if it satisfies the doubling condition

$$w(2B) \leqslant Cw(B) \text{ for any ball } B \tag{1.6}$$

and some constant C.

The classes of weights defined in this work are naturally associated to the decay of the kernel of $\mathscr{I}_\alpha$ and the function "critical radius" ρ related. There is some connection between these classes and a two-weighted version of the classes $A_p^{\rho,\theta}$, introduced in [4] in relation with L^p-norm inequalities for several operators. As usual p' denotes the Hölder's conjugate exponent of p.

DEFINITION 1.1. Given $1 < p \leqslant \infty$, δ and λ real numbers and $N_0 \geqslant 0$, the pair of weights (v,w) belongs to $\mathscr{S}(p,\delta,\lambda,N_0)$ if there exist $C > 0$ such that

$$\Big(\frac{v^{p'}(\theta B)}{|\theta B|^{1-\frac{p'\lambda}{d}}}\Big)^{\frac{1}{p'}} \leqslant C\Big(1+\frac{\theta R}{\rho(x)}\Big)^{N_0}\frac{w(B)}{|B|^{1-\frac{\delta}{d}}}$$

for all $\theta \geqslant 1$, any ball $B = B(x,R)$ and $\theta B = B(x,\theta R)$. The above inequality can be rephrased as

$$\Big(v^{p'}(\theta B)\Big)^{\frac{1}{p'}} \leqslant C\Big(1+\frac{\theta R}{\rho(x)}\Big)^{N_0}\theta^{\frac{d}{p'}-\lambda}\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\delta-\lambda}{d}}}.$$

The class $\mathscr{S}(1,\delta,\lambda,N_0)$ is defined by

$$\sup_{\theta B} v \leqslant C\Big(1+\frac{\theta R}{\rho(x)}\Big)^{N_0}\theta^{-\lambda}\frac{w(B)}{|B|^{1-\frac{\delta-\lambda}{d}}}. \tag{1.7}$$

We also define $\mathscr{S}(p,\delta,\lambda) = \bigcup_{N_0\geqslant 0}\mathscr{S}(p,\delta,\lambda,N_0)$. If (v,v) belongs to $\mathscr{S}(p,\delta,\lambda)$ we simply say that $v \in \mathscr{S}(p,\delta,\lambda)$. Our first result is

THEOREM 1.1. *Let $1 < p < \infty$, $\alpha > 0$ and $\alpha - \frac{d}{p} - \lambda < \delta_0$. If w is doubling (see (1.6)), $(v,w) \in \mathscr{S}((p'r)',\delta,\lambda)$ for some $r > 1$ and $\lambda - \delta \leqslant \alpha - \frac{d}{p}$ then there exists a positive constant C such that for all $f \in L^{p,\infty}(v)$*

$$\frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f(x) - c_B|dx \leqslant CR^{\alpha-\frac{d}{p}+\delta-\lambda}\Big[\frac{f}{v}\Big]_p,\ R < \rho(x_B) \tag{1.8}$$

for any (subcritical) ball $B = B(x_B,R)$, and some positive constant c_B and

$$\frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f(x)|dx \leqslant C\rho(x_B)^{\alpha-\frac{d}{p}+\delta-\lambda}\Big[\frac{f}{v}\Big]_p \tag{1.9}$$

for any critical ball $B = B(x_B,\rho(x_B))$.

In the one-weighted situation, the following corollary of Theorem 1.1 improves the result in [2] since the inequalities therein are deduced for a wider class of weights.

COROLLARY 1.1. *Let $1 < p < \infty$, $\frac{d}{p} \leqslant \alpha$ and $\alpha - \frac{d}{p} - \lambda < \delta_0$. If v is doubling and belongs to $\mathscr{S}(p,\lambda,\lambda)$ then there exists a constant C such that for all $f \in L^{p,\infty}(v)$*

$$\frac{1}{v(B)}\int_B |\mathscr{I}_\alpha f(x) - c_B|dx \leqslant CR^{\alpha-\frac{d}{p}}\Big[\frac{f}{v}\Big]_p, \tag{1.10}$$

for $B = B(x_B,R)$ with $R < \rho(x_B)$, and some positive constant c_B, and

$$\frac{1}{v(B)}\int_B |\mathscr{I}_\alpha f(x)|dx \leqslant C\rho(x_B)^{\alpha-\frac{d}{p}}\Big[\frac{f}{v}\Big]_p \tag{1.11}$$

for any critical ball $B = B(x_B,\rho(x_B))$.

Inequalities (1.8) and (1.9) indicate that the arrival space for $\mathscr{I}_\alpha$ is the following weighted Campanato-type space associated to the Schrödinger operator which was introduced in [2].

DEFINITION 1.2. Given a weight w and $\beta \geqslant 0$ the space $BMO^\beta_{\mathscr{L}}(w)$ is the set of functions f in L^1_{loc} satisfying for any ball $B = B(x,r)$, with $x \in \mathbb{R}^d$ and $r > 0$,

$$\frac{1}{w(B)} \int_B |f - f_B| \leqslant C|B|^{\frac{\beta}{d}}, \quad \text{with } f_B = \frac{1}{|B|}\int_B f, \tag{1.12}$$

and

$$\frac{1}{w(B)} \int_B |f| \leqslant C|B|^{\frac{\beta}{d}}, \quad \text{if } r \geqslant \rho(x). \tag{1.13}$$

Since (1.13) implies (1.12) for $r \geqslant \rho(x)$ then it is enough to consider (1.12) only for radius $r < \rho(x)$. The constants in (1.12) and (1.13) are independent of the choice of B but may depend on f. A norm (up to an identification of functions differing by a constant) in the space $BMO^\beta_{\mathscr{L}}(w)$ is given by the infima of the constants C satisfying (1.12) and (1.13). As in the classical case, the mean value f_B in (1.12) may be replaced by any positive constant c_B depending only on the ball.

In view of the above definition we are able to rephrase Theorem 1.1 and Corollary 1.1 in terms of a continuous mapping. That is,

$$\mathscr{I}_\alpha : L^{p,\infty}(v) \to BMO^{\alpha - \frac{d}{p} + \delta - \lambda}_{\mathscr{L}}(w)$$

and

$$\mathscr{I}_\alpha : L^{p,\infty}(v) \to BMO^{\alpha - \frac{d}{p}}_{\mathscr{L}}(v),$$

with continuity.

In our next theorem we obtain pointwise regularity estimates for $\mathscr{I}_\alpha f$ whenever $f \in BMO^\beta_{\mathscr{L}}(v)$ with β small. Those kind of inequalities characterize weighted Campanato-type spaces of order smaller than 1. Let us consider the function W_β defined by

$$W_\beta(x,r) = \int_{B(x,r)} \frac{w(z)}{|z-x|^{d-\beta}}\, dz \tag{1.14}$$

for $x \in \mathbb{R}^d$, $r > 0$, $\beta > 0$ and $w \in L^1_{loc}$.

THEOREM 1.2. *Let $\alpha > 0$, $\beta \geqslant 0$, $\beta > \lambda$, $\beta - \lambda + \alpha < \delta_0$ and δ and λ be real numbers. If $0 < \beta + \alpha + \delta - \lambda < 1$, w is a doubling weight (see ((1.6))) and $(v,w) \in \mathscr{S}(\infty, \delta, \lambda)$ then there exists a constant C such that for all $f \in BMO^\beta_{\mathscr{L}}(v)$*

$$|\mathscr{I}_\alpha f(x) - \mathscr{I}_\alpha f(y)| \leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(v)} \left(W_{\beta+\alpha+\delta-\lambda}(x,|x-y|) + W_{\beta+\alpha+\delta-\lambda}(y,|x-y|)\right) \tag{1.15}$$

if $|x-y|<\rho(x)$ *and*

$$\frac{1}{w(B(x,\rho(x)))}\int_{B(x,\rho(x))}|\mathscr{I}_\alpha f(x)|dx \leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(v)}\rho(x)^{\beta+\alpha+\delta-\lambda} \tag{1.16}$$

for all $x\in\mathbb{R}^d$ *and* $r>0$.

In the case $v=w$ and $\delta-\lambda=0$ in Theorem 1.2 we recover Theorem 2 in [2].

Notice that the function $W_\beta(x,r)$ defined in (1.14) is finite for all $r>0$ for almost every $x\in\mathbb{R}^d$. It also is increasing as a function of r for any fixed x and if w is doubling (see (1.6)) then W_β is also doubling in the same sense.

The function W_β was first considered in [17] and later also used in [2] to define a Lipschitz-Hölder-type space associated to $\mathscr{L}$. That is, $\Lambda^\beta_{\mathscr{L}}(w)$ is the set of functions f such that

$$|f(x)-f(y)| \leqslant C\Big(W_\beta(x,|x-y|)+W_\beta(y,|x-y|)\Big) \tag{1.17}$$

and

$$|f(x)| \leqslant CW_\beta(x,\rho(x)). \tag{1.18}$$

for almost all x and y in $\mathbb{R}^d$. A (quasi) norm is defined on $\Lambda^\beta_{\mathscr{L}}(w)$ by taking the maximum of the two infima of the constants satisfying (1.17) and (1.18) respectively.

PROPOSITION 1.1. ([2]) *If* $0<\beta<1$ *and* w *satisfies the doubling condition* (1.6) *then* $\Lambda^\beta_{\mathscr{L}}(w)=BMO^\beta_{\mathscr{L}}(w)$ *and their norms are equivalent.*

REMARK 1.1. When proving Proposition 1.1, the authors showed that (1.13) jointly with (1.17) imply (1.18). That is if f satisfy (1.13) and (1.17) simultaneously then $f\in\Lambda^\beta_{\mathscr{L}}(w)$. The fact that w is doubling is essential to obtain the identification between Lipschitz-Hölder and Campanato type spaces in Proposition 1.1. Hence Theorem 1.2 can be rephrased by saying that

$$\mathscr{I}_\alpha : BMO^\beta_{\mathscr{L}}(v)\to\Lambda^{\beta+\alpha+\delta-\lambda}_{\mathscr{L}}(w)=BMO^{\beta+\alpha+\delta-\lambda}_{\mathscr{L}}(w)$$

is continuous, under the hypothesis of that theorem. In the one-weight situation, we obtain

$$\mathscr{I}_\alpha : BMO^\beta_{\mathscr{L}}(v)\to\Lambda^{\beta+\alpha}_{\mathscr{L}}(v)=BMO^{\beta+\alpha}_{\mathscr{L}}(v).$$

This article is organized as follows. In Section 2 we give the preliminary definitions and results. In Section 3 we study properties of the class of weights given in Definition 1.1 and show some examples. Section 4 and Section 5 are devoted to the main lemmas and the proofs of Theorems 1.1 and 1.2 respectively.

Throughout this work, we denote by C a constant that may change from one occurrence to other.

2. Preliminaries

Some important properties of the critical ratii ρ given in (1.5) are shown in the following propositions.

PROPOSITION 2.1. ([36]) *There exist C and $j_0 \geqslant 1$ such that*

$$C^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-j_0} \leqslant \rho(y) \leqslant C\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{j_0}{j_0+1}}$$

for all $x,y\in\mathbb{R}^d$.

PROPOSITION 2.2. ([11]) *There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^d$, so that the family $B_k = B(x_k,\rho(x_k))$, $k\geqslant 1$, satisfies*

1. *$\cup_k B_k = \mathbb{R}^d$.*
2. *There exists N such that for all $k\in\mathbb{N}$, $card\{j:\ 4B_j\cap 4B_k \neq \emptyset\} \leqslant N$.*

The doubling condition on w is a crucial point in the proof given by the authors in [2] of the following proposition.

PROPOSITION 2.3. ([2]) *Given $\beta\geqslant 0$, $w\in D$ (see (1.6)) and $\{x_k\}_{k=1}^{\infty}$ a sequence as in Proposition 2.2, a function f belongs to $BMO^{\beta}_{\mathscr{L}}(w)$ if, and only if, f satisfies (1.12) for any ball B, and*

$$\int_{B(x_k,\rho(x_k))}|f| \leqslant Cw(B(x_k,\rho(x_k)))\,\rho(x_k)^{\beta} \quad \textit{for all } k\geqslant 1. \tag{2.1}$$

The previous result allows us to provide the following characterization of the space $BMO^{\beta}_{\mathscr{L}}(w)$ that in the sequel will be used as it definition.

COROLLARY 2.1. ([2]) *Let $\beta\geqslant 0$ and $w\in D$ (see (1.6)). A function f belongs to $BMO^{\beta}_{\mathscr{L}}(w)$ if, and only if, for some constant C*

$$\frac{1}{w(B)}\int_B|f-f_B| \leqslant CR^{\beta},\ \textit{if } B=B(x,r) \textit{ and } r<\rho(x) \tag{2.2}$$

and

$$\frac{1}{w(B(x,\rho(x)))}\int_{B(x,\rho(x))}|f| \leqslant C\rho(x)^{\beta}. \tag{2.3}$$

If $w\equiv 1$ the atomic decomposition given in [11] and [13] shows that $BMO^{\beta}_{\mathscr{L}}$, $\beta\geqslant 0$, is the dual space of the H^p-space defined in those works. In this setting, the $BMO_{\mathscr{L}}$ space, $\beta=0$, was defined in [10] as a natural substitute of L^{∞} in the context of the semigroup generated by the operator $\mathscr{L}$.

In the case that w is not doubling then inequality (2.1) implies the condition

$$\int_B |f| \leqslant C\,w(cB)\,|B|^{\frac{\beta}{d}}, \quad \text{if } r \geqslant \rho(x), \tag{2.4}$$

for the geometric constant $c = c_0$ in Proposition 2.1 and some $C > 0$. Therefore, if w is not doubling a different space should be defined, $BMO^{\beta}_{c,\mathscr{L}}(w)$, c a positive number, as the one satisfying (2.4) and

$$\int_B |f - f_B| \leqslant C\,w(cB)\,|B|^{\frac{\beta}{d}}, \quad \text{if } r < \rho(x), \tag{2.5}$$

for some constant C. Clearly, $BMO^{\beta}_{1,\mathscr{L}}(w) = BMO^{\beta}_{\mathscr{L}}(w)$. Thus, a different version of Corollary 2.1 can be obtained as follows,

COROLLARY 2.2. *Let $\beta \geqslant 0$ and w a weight. If f belongs to $BMO^{\beta}_{c,\mathscr{L}}(w)$ then for some constant C it satisfies*

$$\int_B |f - f_B| \leqslant C\,w(cB)\,R^{\beta} \quad \textit{if } B = B(x,r) \ \textit{ and } r < \rho(x), \tag{2.6}$$

and

$$\int_{B(x,\rho(x))} |f| \leqslant C w(B(x, c\rho(x)))\,\rho(x)^{\beta} \quad \textit{for all } x \in \mathbb{R}^d. \tag{2.7}$$

Reciprocally, if f satisfies inequalities (2.6) and (2.7) then f belongs to $BMO^{\beta}_{\tilde{c},\mathscr{L}}(w)$ where $\tilde{c} \geqslant c$.

Hence, by Corollary 2.2 if w is not doubling it is still possible to obtain a weaker version of Theorem 1.1 and Corollary 1.1, with $BMO^{\alpha-\frac{d}{p}+\delta-\lambda}_{c,\mathscr{L}}(w)$ as arrival space.

In the remaining part of this section some useful lemmas related to the kernel K_α of $\mathscr{I}_\alpha$ will be stated and given the references to their proofs. That kernel is given by the formula

$$K_\alpha(x,y) = \int_0^{\infty} k_t(x,y)\,t^{\alpha/2}\,\frac{dt}{t},$$

where k_t is the kernel of the operator $e^{-t\mathscr{L}}$ $(t > 0)$.

LEMMA 2.1. ([23]) *Given $N > 0$ there exists a constant $C = C_N$ such that for all x and y in $\mathbb{R}^d$,*

$$k_t(x,y) \leqslant C t^{-\frac{d}{2}}\, e^{-\frac{|x-y|^2}{Ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

As consequence of the above inequality it follows that

$$K_\alpha(x,y) \ \leqslant \ \frac{C}{|x-y|^{d-\alpha}} \tag{2.8}$$

for all x and y in $\mathbb{R}^d$.

LEMMA 2.2. ([13]) *Given $N>0$ and $0<\nu<\delta_0$, there exists a constant $C=C_N$ such that*

$$|k_t(x,y)-k_t(x_0,y)| \leqslant C\Big(\frac{|x-x_0|}{\sqrt{t}}\Big)^{\nu} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Ct}} \Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N}$$

for all x, y and x_0 in $\mathbb{R}^d$ with $|x-x_0|<\sqrt{t}$.

We recall that a function ψ is said to be *rapidly decaying* (see [12]) if for each $N>0$ there exists a constant C_N such that $|\psi(x)| \leqslant C_N(1+|x|)^{-N}$. Its dilation is defined by $\psi_t(x)=\frac{1}{t^{\frac{d}{2}}}\psi\Big(\frac{x}{\sqrt{t}}\Big)$ for $t>0$.

Some estimates on the function $q_t(x,y)=k_t(x,y)-\tilde{k}_t(x,y)$, where $\tilde{k}_t$ is the kernel of the classical heat operator $e^{-t\Delta}$, will be useful later.

LEMMA 2.3. ([12]) *There exist a rapidly decaying no negative function ψ, $0<\nu<2-\frac{d}{q_0}$ and $C>0$ such that for $x,y\in\mathbb{R}^d$ and $t>0$*

$$|q_t(x,y)| \leqslant C\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu} \psi_t(x-y).$$

LEMMA 2.4. ([12]) *For all $0<\nu<\delta_0$ there exists a rapidly decaying function ψ and a constant $C>0$ such that $|q_t(x,y+h)-q_t(x,y)| \leqslant \Big(\frac{|h|}{\rho(x)}\Big)^{\nu}\psi_t(x-y)$ for x, y in $\mathbb{R}^d$, $t>0$ and $|h|<\min(C\rho(y),\frac{|x-y|}{4})$.*

3. Properties and examples of the class of weights

LEMMA 3.1. *The classes $\mathscr{S}(p,\delta,\lambda)$ are increasing in p for $1\leqslant p\leqslant\infty$ and δ and λ real numbers. That is, $\mathscr{S}(p,\delta,\lambda)\subseteq\mathscr{S}(\tau p,\delta,\lambda)\subseteq\mathscr{S}(\infty,\delta,\lambda)$ for all $\tau>1$ and $1\leqslant p<\infty$.*

Proof. Let $B=B(x,t)$. By Hölder's inequality and Definition 1.1 there exists $C>0$ and $N_0\geqslant 0$ such that, if $p>1$,

$$\Big(\frac{v^{(\tau p)'}(\theta B)}{|\theta B|}\Big)^{\frac{1}{(\tau p)'}} \leqslant \Big(\frac{v^{p'}(\theta B)}{|\theta B|}\Big)^{\frac{1}{p'}} \leqslant C\Big(1+\frac{\theta t}{\rho(x)}\Big)^{N_0}\theta^{-\lambda}\frac{w(B)}{|B|^{1-\frac{\delta-\lambda}{d}}}.$$

For $p=1$ the proof follows in the same way by replacing the second term above by $\sup_{\theta B} v$. That is,

$$\Big(\frac{v^{\tau'}(\theta B)}{|\theta B|}\Big)^{\frac{1}{\tau'}} \leqslant C\sup_{\theta B} v \leqslant C\Big(1+\frac{\theta t}{\rho(x)}\Big)^{N_0}\theta^{-\lambda}\frac{w(B)}{|B|^{1-\frac{\delta-\lambda}{d}}}.$$

for $1\leqslant p<\infty$. Analogously, $\mathscr{S}(p,\delta,\lambda)\subseteq\mathscr{S}(\infty,\delta,\lambda)$. □

In order to show a more precise relationship between the class of weights defined in Definition 1.1 and the classes introduced in [17] and [34] we prove the next lemma.

LEMMA 3.2. *Given* $1 < p < \infty$*,* λ*,* δ *real numbers and* $N_0 \geqslant 0$*, if* $(v,w) \in \mathscr{S}(p,\delta,\lambda,N_0)$ *and* $\xi + \frac{d}{p} + \lambda > 0$ *then there exists a constant* $C > 0$ *such that*

$$|B|^{1+\frac{\xi}{d}}\Big(\int_{\mathbb{R}^d - B} \frac{v^{p'}(y)}{|x-y|^{(d+\xi)p'}}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-Np'} dy\Big)^{\frac{1}{p'}} \leqslant C\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\gamma}{d}}}, \tag{3.1}$$

and

$$\Big(v^{p'}(B)\Big)^{\frac{1}{p'}}\Big(1+\frac{|B|^{\frac{1}{d}}}{\rho(x)}\Big)^{-N} \leqslant C\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\gamma}{d}}}, \tag{3.2}$$

where $\gamma = \delta - \lambda$*, and* $N \geqslant N_0$ *and both inequalities hold for any ball* $B = B(x,R)$*. Reciprocally, if* (3.1) *and* (3.2) *hold for some* ξ *and* γ *real numbers and* $N \geqslant 0$ *then* $(v,w) \in \mathscr{S}(p,\gamma - \frac{d}{p} - \xi, -\frac{d}{p} - \xi, N)$*.*

Proof. Inequality (3.2) is a direct application of Definition 1.1 when $\theta = 1$. On the other hand, given a ball B of radius t, by a dyadic decomposition and Definition 1.1 we get

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^d - B} \frac{v^{p'}(y)}{|x-y|^{(d+\xi)p'}}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-Np'} dy\Big)^{\frac{1}{p'}}\\
&\leqslant C\Big(\sum_{k=0}^{\infty}\frac{1}{(2^k t)^{(d+\xi)p'}}v^{p'}(B(x,2^k t))\Big(1+\frac{2^k t}{\rho(x)}\Big)^{-Np'}\Big)^{\frac{1}{p'}}\\
&\leqslant C\frac{1}{|B|^{1+\frac{\xi}{d}}}\Big(\sum_{k=0}^{\infty}2^{-k(d+\xi)p'}2^{k(d-\lambda p')}\Big)^{\frac{1}{p'}}\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\delta-\lambda}{d}}}\\
&\leqslant C\frac{w(B)}{|B|^{1+\frac{1}{p}+\frac{\xi}{d}-\frac{\delta-\lambda}{d}}}\Big(\sum_{k=0}^{\infty}2^{-k(\xi+\frac{d}{p}+\lambda)p'}\Big)^{\frac{1}{p'}}\\
&\leqslant C\frac{w(B)}{|B|^{1+\frac{1}{p}+\frac{\xi}{d}-\frac{\delta-\lambda}{d}}}
\end{aligned}$$

if $\xi + \frac{d}{p} + \lambda > 0$.

Reciprocally, (3.1) and (3.2) jointly are equivalent to the inequality

$$\begin{aligned}
&|B|^{1+\frac{\xi}{d}}\Big(\int_{\mathbb{R}^d} v^{p'}(y)\Big[|x-y|^{(d+\xi)p'}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{Np'} + |B|^{(1+\frac{\xi}{d})p'}\Big(1+\frac{|B|^{\frac{1}{d}}}{\rho(x)}\Big)^{Np'}\Big]^{-1} dy\Big)^{\frac{1}{p'}}\\
&\leqslant C\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\gamma}{d}}}.
\end{aligned}$$

Moreover, we easily get that

$$|B|^{1+\frac{\xi}{d}}\left(\int_{\mathbb{R}^d} v^{p'}(y)\left[|x-y|^{(d+\xi)p'}\left(1+\frac{|x-y|}{\rho(x)}\right)^{Np'}+|B|^{(1+\frac{\xi}{d})p'}\left(1+\frac{|B|^{\frac{1}{d}}}{\rho(x)}\right)^{Np'}\right]^{-1}dy\right)^{\frac{1}{p'}}$$
$$\geqslant B|^{1+\frac{\xi}{d}}\frac{(v^{p'}(\theta B))^{\frac{1}{p'}}}{|\theta B|^{1+\frac{\xi}{d}}\left(1+\frac{\theta R}{\rho(x)}\right)^{N}}.$$

Then

$$|B|^{1+\frac{\xi}{d}}\frac{(v^{p'}(\theta B))^{\frac{1}{p'}}}{|\theta B|^{1+\frac{\xi}{d}}\left(1+\frac{\theta R}{\rho(x)}\right)^{N}}\leqslant C\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\gamma}{d}}}.$$

That is, $(v^{p'}(\theta B))^{\frac{1}{p'}}\leqslant C\theta^{d+\xi}\left(1+\frac{\theta R}{\rho(x)}\right)^{N}\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\gamma}{d}}}$. Now setting $\frac{d}{p'}-\lambda=d+\xi$ and $\delta-\lambda=\gamma$, we obtain that $(v,w)\in\mathscr{S}(p,\delta,\lambda,N)$ with $\delta=\gamma-\frac{d}{p}-\xi$ and $\lambda=-\frac{d}{p}-\xi$. $\square$

The above lemma allows us to compare the family of weights defined in this work with other known classes. For example, taking $N_0=0$, $\alpha=\delta-\lambda$, $-(\frac{d}{p}+\lambda)<1-\alpha$ and p fixed in our class we obtain the set of weights $H(p,\alpha,2\alpha-\frac{d}{p}-d)$ in [34] and if, in addition, we set $\delta-\lambda=0$ and consider the family of one weights $v=w$ then we obtain the class in [17].

3.1. The case $\delta=\lambda$

This special case of classes of weights displays significant features that will be described in this section. Let us first define an extension of the class $A_p^{\rho,\theta}$ introduced in [4] to a family of pair of weights in the following way. The pair (w,v) belongs to $A_p^{\rho,N}$, $N\geqslant 0$, if there exists a constant C such that

$$(w(B))^{\frac{1}{p}}\left(v^{-\frac{1}{p-1}}(B)\right)^{\frac{1}{p'}}\leqslant C|B|\left(1+\frac{r}{\rho(x)}\right)^{N}$$

for every ball $B=B(x,r)$ and $1<p<\infty$. In the case $p=1$ the pair (w,v) satisfies inequality

$$w(B)\leqslant C|B|\inf_B v\left(1+\frac{r}{\rho(x)}\right)^{N}.$$

PROPOSITION 3.1. *Let $1<q<\infty$. If $(w^{q'},v^{q'})\in A_1^{\rho,N}$ then $(w,v)\in\mathscr{S}(q,0,0,\frac{N}{q'})$.*

Proof. Given $\theta \geqslant 1$ and $B = B(x,r)$, statement $(w^{q'}, v^{q'}) \in A_1^{\rho,N}$ implies that

$$
\begin{aligned}
(w^{q'}(\theta B))^{\frac{1}{q'}} \left(1+\frac{\theta r}{\rho(x)}\right)^{-\frac{N}{q'}} &\leqslant C|\theta B|^{\frac{1}{q'}} \inf_{\theta B} v \leqslant C\theta^{\frac{d}{q'}} |B|^{\frac{1}{q'}} \inf_{B} v \\
&\leqslant C\theta^{\frac{d}{q'}} |B|^{\frac{1}{q'}} \frac{v(B)}{|B|} \leqslant C\theta^{\frac{d}{q'}} \frac{v(B)}{|B|^{\frac{1}{q}}},
\end{aligned}
$$

which is the desired inequality. $\square$

PROPOSITION 3.2. *Let* $1 < q < \infty$. *If* $w \in D_\eta$ *and* $(v^{q'}, w^{q'}) \in A_{q'+1}^{\rho,N}$ *then* $(v,w) \in \mathscr{S}(q, -\eta, -\eta, N(1+\frac{1}{q'}))$.

Proof. The hypothesis $(v^{q'}, w^{q'}) \in A_{q'+1}^{\rho,N}$ implies that

$$
v^{q'}(\theta B)\left(w^{-1}(\theta B)\right)^{q'} \leqslant C|\theta B|^{q'+1}\left(1+\frac{\theta r}{\rho(x_{\theta B})}\right)^{N(q'+1)}.
$$

Then Hölder's inequality, the doubling condition and the fact $|\theta B| \geqslant |B|$ show that

$$
\begin{aligned}
(v^{q'}(\theta B))^{\frac{1}{q'}} \left(1+\frac{\theta r}{\rho(x_{\theta B})}\right)^{-N(1+\frac{1}{q'})} &\leqslant C|\theta B|^{\frac{1}{q'}} \frac{|\theta B|}{w^{-1}(\theta B)} \\
&\leqslant C|\theta B|^{\frac{1}{q'}} \frac{w(\theta B)}{|\theta B|} \\
&\leqslant C\theta^{\frac{d}{q'}+\eta} \frac{w(B)}{|B|^{\frac{1}{q}}}
\end{aligned}
$$

which proves that $(v,w) \in \mathscr{S}(q, -\eta, -\eta, N(1+\frac{1}{q'}))$. $\square$

The one weight case $v \in \mathscr{S}(p, \lambda, \lambda)$ satisfies special properties. The case $N_0 = 0$ was defined in [2] in connection with the Schrödinger operator and in [17] related to the Laplacian operator. One of the features of this class of weights is a Reverse-Hölder-type inequality. Thereafter, we will be able to recover the results in [2] for $N_0 = 0$.

DEFINITION 3.1. Given $1 \leqslant p < \infty$ a weight v satisfies the Reverse-Hölder type inequality of order p, and say $v \in RH_p(\rho)$ if there exist $C > 0$ and $N_0 \geqslant 0$ such that for any ball $B = B(x,R)$,

$$
\left(\frac{v^p(B)}{|B|}\right)^{\frac{1}{p}} \leqslant C\left(1+\frac{R}{\rho(x)}\right)^{N_0} \frac{v(B)}{|B|}. \tag{3.3}
$$

We say that $v \in RH_\infty(\rho)$, if there exist $C > 0$ and $N_0 \geqslant 0$ such that for any ball $B = B(x,R)$,

$$
\sup_B v \leqslant C\left(1+\frac{R}{\rho(x)}\right)^{N_0} \frac{v(B)}{|B|}. \tag{3.4}
$$

LEMMA 3.3. *Let $1 \leqslant p \leqslant \infty$ and λ be a real number. If $v \in \mathscr{S}(p,\lambda,\lambda)$ then*

1. $v \in RH_{p'}(\rho)$.

2. *v is $\mathscr{L}$-doubling, that is, there exist $\eta_1 > 0$, $C > 0$ and N_0 big enough, such that*

$$v(\theta B) \leqslant C\theta^{\eta_1}\left(1+\frac{R}{\rho(x)}\right)^{N_0} v(B)$$

for any ball $B = B(x,R)$.

3. *$v^{p'}$ (if $p' < \infty$) or $\sup v$ (if $p' = \infty$) are $\mathscr{L}$-doubling, that is, there exist $\eta_2 > 0$, $C > 0$ and N_0 big enough, such that*

$$v^{p'}(\theta B) \leqslant C\theta^{\eta_2}\left(1+\frac{R}{\rho(x)}\right)^{N_0 p'} v^{p'}(B) \quad \text{if } p' < \infty \tag{3.5}$$

or

$$\sup_{\theta B} v \leqslant C\theta^{\eta_2}\left(1+\frac{R}{\rho(x)}\right)^{N_0} \sup_B v \qquad \text{if } p' = \infty$$

for any ball $B = B(x,R)$.

Reciprocally, if $v \in RH_{p'}(\rho)$ and it is $\mathscr{L}$-doubling then there exists a real number λ such that $v \in \mathscr{S}(p,\lambda,\lambda)$.

Proof. The first item is an immediate consequence of Definition 1.1 in the case $\delta = \lambda$ and $\theta = 1$. For the second and third item we apply first Hölder's inequality, the Definition 1.1 for some $N_0 \geqslant 0$ and $\delta = \lambda$ and, again, Hölder's inequality, to get

$$\begin{aligned}\frac{v(\theta B)}{|\theta B|} &\leqslant \left(\frac{v^{p'}(\theta B)}{|\theta B|}\right)^{\frac{1}{p'}} \leqslant C\theta^{-\lambda+N_0}\left(1+\frac{R}{\rho(x)}\right)^{N_0}\frac{v(B)}{|B|} \\ &\leqslant C\theta^{-\lambda+N_0}\left(1+\frac{R}{\rho(x)}\right)^{N_0}\left(\frac{v^{p'}(B)}{|B|}\right)^{\frac{1}{p'}}.\end{aligned}$$

Hence, taking on one side the first and third term and, on the other, the second and fourth term, we obtain

$$v(\theta B) \leqslant C\theta^{d-\lambda+N_0}\left(1+\frac{R}{\rho(x)}\right)^{N_0} v(B) \tag{3.6}$$

and

$$v^{p'}(\theta B) \leqslant C\theta^{(\frac{d}{p'}-\lambda+N_0)p'}\left(1+\frac{R}{\rho(x)}\right)^{N_0 p'} v^{p'}(B) \qquad \text{if } p' < \infty.$$

On the other hand taking $\sup v$ in (3.6) it follows the case $p' = \infty$.

Reciprocally, if $v \in RH_{p'}(\rho)$ and it is $\mathscr{L}$-doubling then there exists a constant C and a real number η such that

$$\begin{aligned}\Big(\frac{v^{p'}(\theta B)}{|\theta B|}\Big)^{\frac{1}{p'}} &\leqslant C\Big(1+\frac{\theta R}{\rho(x)}\Big)^{N_0}\frac{v(\theta B)}{|\theta B|}\\ &\leqslant C\Big(1+\frac{\theta R}{\rho(x)}\Big)^{N_0}\theta^{\eta-d}\Big(1+\frac{R}{\rho(x)}\Big)^{N_1}\frac{v(B)}{|B|}\\ &\leqslant C\Big(1+\frac{\theta R}{\rho(x)}\Big)^{N_0+N_1}\theta^{\eta-d}\frac{v(B)}{|B|}.\end{aligned}$$

That is,

$$\Big(v^{p'}(\theta B)\Big)^{\frac{1}{p'}} \leqslant C\theta^{\frac{d}{p'}+\eta-d}\Big(1+\frac{\theta R}{\rho(x)}\Big)^{N_0+N_1}\frac{v(B)}{|B|^{\frac{1}{p}}}$$

which implies that $v \in \mathscr{S}(p,d-\eta,d-\eta)$. Analogously, if $v \in RH_\infty(\rho)$ and v is $\mathscr{L}$-doubling then $v \in \mathscr{S}(\infty,d-\eta,d-\eta)$. $\square$

The next lemma is the fundamental key for proving in the one–weight setting that the class $\mathscr{S}(p,\lambda,\lambda)$ is open to the left in p.

LEMMA 3.4. *Let $1<q\leqslant\infty$ and λ be a real number. If $v \in \mathscr{S}(q,\lambda,\lambda)$ then there exist $\tau_0>1$ and $K_0 \geqslant 0$ such that*

$$\Big(\frac{v^{\tau q'}(B)}{|B|}\Big)^{\frac{1}{\tau q'}} \leqslant C\Big(1+\frac{R}{\rho(x)}\Big)^{K_0}\Big(\frac{v^{q'}(B)}{|B|}\Big)^{\frac{1}{q'}}$$

for $1\leqslant\tau\leqslant\tau_0$ and any ball $B=B(x,r)$.

Proof. By Lemma 3.3 and Definition 3.3 we get $\Big(\frac{v^{q'}(B)}{|B|}\Big)^{\frac{1}{q'}} \leqslant C\frac{v(B)}{|B|}$ for any ball $B=B(x,R)$ such that $R\leqslant\rho(x)$. In this situation, the proof in [16], page 268, shows that there is $\tau_0>1$ and a constant C_1 such that if $1\leqslant\tau\leqslant\tau_0$ then

$$\Big(\frac{v^{\tau q'}(B)}{|B|}\Big)^{\frac{1}{\tau q'}} \leqslant C_1\Big(\frac{v^{q'}(B)}{|B|}\Big)^{\frac{1}{q'}}. \tag{3.7}$$

On the other hand, in the case $R\geqslant\rho(x)$ let us denote $\mathscr{F}=\{j : B_j\cap B\neq\emptyset\}$ with $B_j=B(x_j,\rho(x_j))$ and $\{x_j\}_{j\in\mathbb{N}}$ the sequence in Proposition 2.2. Using Proposition 2.1, if $j\in\mathscr{F}$ and $R\geqslant\rho(x)$ then

$$\rho(x_j)\leqslant C\rho(x)\Big(1+\frac{R}{\rho(x)}\Big)^{\frac{j_0}{j_0+1}} \leqslant C\rho(x)\Big(1+\frac{R}{\rho(x)}\Big) \leqslant CR\Big(1+\frac{R}{\rho(x)}\Big)$$

and, thus, $\bigcup_{j\in\mathscr{F}}B_j\subset cB$, with $c=4C\Big(1+\frac{R}{\rho(x)}\Big)$. By Proposition 2.1, if $j\in\mathscr{F}$ then

$$\begin{aligned}C\rho(x_j) &\geqslant \rho(x)\Big(1+\frac{|x_j-x|}{\rho(x)}\Big)^{-j_0} \geqslant \rho(x)\Big(1+\frac{cR}{\rho(x)}\Big)^{-j_0}\\ &\geqslant \frac{1}{\tilde{C}}\rho(x)\Big(1+\frac{R}{\rho(x)}\Big)^{-j_0}\Big(1+\frac{R}{\rho(x)}\Big)^{-j_0} \geqslant \frac{1}{\tilde{C}}\rho(x)\Big(1+\frac{R}{\rho(x)}\Big)^{-2j_0}.\end{aligned} \tag{3.8}$$

Hence, by (3.7), (3.8) and (3.5), we obtain

$$\begin{aligned}
\Big(\frac{1}{|B|}v^{\tau q'}(B)\Big)^{\frac{1}{\tau q'}} &\leqslant \Big(\frac{1}{|B|}\sum_{j\in\mathscr{F}} v^{\tau q'}(B_j)\Big)^{\frac{1}{\tau q'}} \leqslant C\Big(\frac{1}{|B|}\sum_{j\in\mathscr{F}}\frac{v^{q'}(B_j)^{\tau}}{|B_j|^{\tau-1}}\Big)^{\frac{1}{\tau q'}}\\
&\leqslant C\Big(\frac{1}{|B|\,\rho(x)^{d(\tau-1)}}\Big(1+\frac{R}{\rho(x)}\Big)^{2j_0 d(\tau-1)}\sum_{j\in\mathscr{F}} v^{q'}(B_j)^{\tau}\Big)^{\frac{1}{\tau q'}}\\
&\leqslant C\Big(\frac{1}{|B|R^{d(\tau-1)}}\Big(1+\frac{R}{\rho(x)}\Big)^{(2j_0+1)d(\tau)}\Big)^{\frac{1}{\tau q'}}\Big(\sum_{j\in\mathscr{F}} v^{q'}(B_j)^{\tau}\Big)^{\frac{1}{\tau q'}}\\
&\leqslant C\Big(1+\frac{R}{\rho(x)}\Big)^{3j_0\frac{d}{q'}}\Big(\frac{1}{|B|}\sum_{j\in\mathscr{F}} v^{q'}(B_j)\Big)^{\frac{1}{q'}}\\
&\leqslant C\Big(1+\frac{R}{\rho(x)}\Big)^{3j_0\frac{d}{q'}}\Big(\frac{1}{|B|}\int_{cB}(\sum_{j\in\mathscr{F}}\chi_{B_j})v^{q'}\Big)^{\frac{1}{q'}}\\
&\leqslant C\Big(1+\frac{R}{\rho(x)}\Big)^{3j_0\frac{d}{q'}}\Big(\frac{1}{|B|}v^{q'}(cB)\Big)^{\frac{1}{q'}}\\
&\leqslant C\Big(1+\frac{R}{\rho(x)}\Big)^{3j_0\frac{d}{q'}+N_0}c^{\eta_2}\Big(\frac{1}{|B|}v^{q'}(B)\Big)^{\frac{1}{q'}}\\
&\leqslant C\Big(1+\frac{R}{\rho(x)}\Big)^{K_0}\Big(\frac{1}{|B|}v^{q'}(B)\Big)^{\frac{1}{q'}},
\end{aligned}$$

with $K_0 = 3j_0\frac{d}{q'}+N_0+\eta_2$, and N_0 and η_2 the exponents in (3.5). $\square$

We are now able to prove the openness result we mentioned before. This result is central in the proof of lemmas and theorems in the one-weight situation.

LEMMA 3.5. *Let $1<p<\infty$ and λ be a real number. If $v\in\mathscr{S}(p,\lambda,\lambda)$ then there exists $1<p_0<p$ such that $v\in\mathscr{S}(q,\lambda,\lambda)$ for $p_0<q\leqslant p$.*

Proof. Let us choose τ_0 is as in Lemma 3.4 and set $p_0=(\tau_0 p')'<p$. Note that if $p_0<q\leqslant p$ then $q=(\tau p')'$ for some $1\leqslant\tau\leqslant\tau_0$. Now using Lemma 3.4 and that $v\in\mathscr{S}(p,\lambda,\lambda)$ it follows that

$$\Big(\frac{v^{q'}(\theta B)}{|\theta B|}\Big)^{\frac{1}{q'}} \leqslant C\Big(1+\frac{\theta R}{\rho(x)}\Big)^{K_0}\Big(\frac{v^{p'}(\theta B)}{|\theta B|}\Big)^{\frac{1}{p'}} \leqslant C\Big(1+\frac{\theta R}{\rho(x)}\Big)^{K_0+N_0}\theta^{-\lambda}\,\frac{v(B)}{|B|}$$

for every $\theta\geqslant 1$. $\square$

In the remaining part of this section we show some examples of pairs of weights which belong to the classes defined in this work.

EXAMPLE 3.1. Let us consider the pairs of potential weights $(|x|^{-\varepsilon},|x|^{-\beta})$ and let us analize the values of ε and β that allow this pair to belongs to the class of weights defined in this work. Assuming, for example, that $\rho=1$ then the pair $v(x)=|x|^{-\varepsilon}$ and $w(x)=|x|^{-\beta}$ belongs to the class $\mathscr{S}(p,\delta,\lambda,N_0)$, $1<p<\infty$, if and only if $\beta\leqslant\varepsilon<\frac{d}{p'}$, $\delta\leqslant\min(0,\beta)$ and $N_0\geqslant\lambda-\delta+\beta-\varepsilon\geqslant 0$.

In fact, condition $\varepsilon < \frac{d}{p'}$ is equivalent to the local integrability of $v(x) = |x|^{-\varepsilon}$. Moreover,

$$\Big(\frac{v^{p'}(B(x,r))}{|B(x,r)|}\Big)^{\frac{1}{p'}} \approx \begin{cases} |x|^{-\varepsilon} & \text{if } |x| \geqslant 2r \\ r^{-\varepsilon} & \text{if } |x| < 2r. \end{cases} \tag{3.9}$$

We want to prove inequality

$$\Big(\frac{v^{p'}(B(x,\theta t))}{|B(x,\theta t)|}\Big)^{\frac{1}{p'}} \Big(1+\theta t\Big)^{-N_0} \leqslant C \frac{w(B(x,t))}{|B(x,t)|^{1-\frac{\delta-\lambda}{d}}} \tag{3.10}$$

for all $\theta \geqslant 1$ but, by (3.9), it is equivalent to the following three inequalities

$$\begin{aligned} |x|^{\beta-\varepsilon} &\leqslant C\theta^{\lambda} t^{\delta-\lambda}\Big(1+\theta t\Big)^{N_0} && \text{if} \quad \theta t < \frac{|x|}{2}, \\ |x|^{\beta} &\leqslant C\theta^{\varepsilon+\lambda} t^{\varepsilon+\delta-\lambda}\Big(1+\theta t\Big)^{N_0} && \text{if} \quad t < \frac{|x|}{2} \leqslant \theta t \\ 1 &\leqslant C\theta^{\varepsilon+\lambda} t^{\varepsilon-\beta+\delta-\lambda}\Big(1+\theta t\Big)^{N_0} && \text{if} \quad \frac{|x|}{2} \leqslant t. \end{aligned}$$

However, a careful analysis of the behavior of θ, t and $|x|$ on each region leads to the above inequalities only if $\delta - \lambda \leqslant \beta - \varepsilon \leqslant 0$, $\delta \leqslant \min(0,\beta)$ and $N_0 \geqslant \beta - \varepsilon - \delta + \lambda$. Reciprocally, these conditions are sufficient to prove the above inequalities.

4. Technical Lemmas and proof of Theorem 1.1

The next lemma gives an estimate for the mean value of order $q < p$ on any ball for functions in $L^{p,\infty}(v)$ and it is a fundamental inequality used in the remaining lemmas of this section.

LEMMA 4.1. *Given $1 < q < p < \infty$ and a weight v, there exists a constant C such that*

$$\Big(\int_B \Big(\frac{|f(x)|}{v(x)}\Big)^q dx\Big)^{\frac{1}{q}} \leqslant C|B|^{\frac{1}{q}-\frac{1}{p}} \Big[\frac{f}{v}\Big]_p$$

for any ball B and $f \in L^{p,\infty}(v)$.

Proof. If $q < p$ then for $a = \dfrac{[\frac{f}{v}]_p}{|B|^{\frac{1}{p}}}$ we get the statement as follow

$$\begin{aligned} \int_B \Big(\frac{|f(x)|}{v(x)}\Big)^q dx &= q\Big(\int_0^a + \int_a^\infty\Big) t^{q-1} |B \cap \Big\{\frac{|f|}{v} > t\Big\}| dt \\ &\leqslant |B| a^q + q\Big[\frac{f}{v}\Big]_p^p \int_a^\infty t^{q-p-1} dt \leqslant |B| a^q + \frac{q}{p-q}\Big[\frac{f}{v}\Big]_p^p a^{q-p}. \qquad \square \end{aligned}$$

REMARK 4.1. Condition $q<p$ is crucial to get the statement in Lemma 4.1 since it ensures integrability of the distribution function. Therefore in the foregoing lemmas it will be required that $(v,w)\in\mathscr{S}(q,\delta,\lambda)$ for some $q<p$ when $v\neq w$ or $\delta\neq\lambda$. Nevertheless, in the case $v=w$ and $\delta=\lambda$ by Lemma 3.5 it will be enough to require that $v\in\mathscr{S}(p,\lambda,\lambda)$ since in this case it follows that $v\in\mathscr{S}(q,\lambda,\lambda)$ for all $q<p$ close enough to p. Hence, in the one-weight situation Lemma 4.1 will still apply.

LEMMA 4.2. *Given $1<q<p<\infty$ and λ and δ real numbers if $(v,w)\in\mathscr{S}(q,\delta,\lambda)$ then there exist a positive constant C and $N_0\geqslant 0$ such that for all $f\in L^{p,\infty}(v)$*

$$\int_{\theta B}|f(x)|dx\leqslant C\theta^{\frac{d}{p'}-\lambda}\left(1+\frac{\theta R}{\rho(x)}\right)^{N_0}\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\delta-\lambda}{d}}}\left[\frac{f}{v}\right]_p$$

for all $\theta\geqslant 1$ and every ball $B=B(x_B,R)$. If $v=w\in\mathscr{S}(p,\lambda,\lambda)$, i.e. $\delta=\lambda$, then the above inequality also holds.

Proof. By Hölder's inequality and Lemma 4.1, if $(v,w)\in\mathscr{S}(q,\delta,\lambda)$ for some $q<p$ or $v=w\in\mathscr{S}(p,\lambda,\lambda)$ and $q<p$, close enough to p then, for some $C>0$ and $N_0\geqslant 0$,

$$\begin{aligned}\int_{\theta B}|f(x)|\,dx&\leqslant\left(\int_{\theta B}v^{q'}(x)\,dx\right)^{\frac{1}{q'}}\left(\int_{\theta B}\left(\frac{|f(x)|}{v(x)}\right)^{q}dx\right)^{\frac{1}{q}}\\&\leqslant C\left(1+\frac{\theta R}{\rho(x_B)}\right)^{N_0}\theta^{\frac{d}{q'}-\lambda}\frac{w(B)}{|B|^{\frac{1}{q}-\frac{\delta-\lambda}{d}}}\left(|\theta B|^{\frac{1}{q}-\frac{1}{p}}\left[\frac{f}{v}\right]_p\right)\\&\leqslant C\theta^{\frac{d}{p'}-\lambda}\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\delta-\lambda}{d}}}\left(1+\frac{\theta R}{\rho(x_B)}\right)^{N_0}\left[\frac{f}{v}\right]_p.\qquad\square\end{aligned}$$

To prove Theorem 1.1 we need estimates for the fractional integral of the local and global parts of a function $f\in L^{p,\infty}(v)$. The following two lemmas give us those estimates.

LEMMA 4.3. *Given $1<q<p<\infty$, λ and δ real numbers and $\alpha>0$, if $(v,w)\in\mathscr{S}(q,\delta,\lambda)$ then there exists $N_0\geqslant 0$ and C such that for all $f\in L^{p,\infty}(v)$*

$$\frac{1}{w(B)}\int_B\mathscr{I}_\alpha(|f|\chi_{2B})(x)dx\leqslant C\left(1+\frac{R}{\rho(x_B)}\right)^{N_0}|B|^{\frac{\alpha}{d}-\frac{1}{p}+\frac{\delta-\lambda}{d}}\left[\frac{f}{v}\right]_p$$

for any ball $B=B(x_B,R)$. If $v=w\in\mathscr{S}(p,\lambda,\lambda)$, i.e. $\delta=\lambda$, then the above inequality also holds.

Proof. By (2.8), Fubini's Theorem, $\alpha > 0$ and Lemma 4.2 applied to $\theta = 2$ there exists $N_0 \geqslant 0$ such that

$$\begin{aligned}
\frac{1}{w(B)}\int_B \mathscr{I}_\alpha(|f|\chi_{2B})(x)dx &= \frac{1}{w(B)}\int_B\int_{2B} K_\alpha(x,y)|f(y)|dy\,dx \\
&\leqslant C\frac{1}{w(B)}\int_B\int_{2B}\frac{|f(y)|}{|x-y|^{d-\alpha}}dy\,dx \\
&\leqslant C\frac{1}{w(B)}\int_{2B}|f(y)|\int_{B(y,2R)}\frac{1}{|x-y|^{d-\alpha}}dx\,dy \\
&\leqslant C\frac{|B|^{\frac{\alpha}{d}}}{w(B)}\int_{2B}|f(y)|dy \\
&\leqslant C\Big(1+\frac{R}{\rho(x_B)}\Big)^{N_0}|B|^{\frac{\alpha}{d}-\frac{1}{p}+\frac{\delta-\lambda}{d}}\Big[\frac{f}{v}\Big]_p. \qquad \square
\end{aligned}$$

LEMMA 4.4. *Given $1 < q < p < \infty$ and λ and δ real numbers, if $(v,w) \in \mathscr{S}(q,\delta,\lambda)$ then there exist positive constants C and $N_0 \geqslant 0$ such that for all $f \in L^{p,\infty}(v)$*

$$\int_{\mathbb{R}^d\setminus 2B}\frac{|f(y)|}{|x-y|^{d+m}}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-N}dy \leqslant C\frac{w(B)}{|B|^{1+\frac{1}{p}+\frac{m+\lambda-\delta}{d}}}\Big[\frac{f}{v}\Big]_p$$

for any constant m such that $m+\lambda+\frac{d}{p} > 0$, $N \geqslant N_0$ and any ball $B = B(x_B,R)$. Moreover, if $v = w \in \mathscr{S}(p,\lambda,\lambda)$, i.e. $\delta = \lambda$, then the above inequality also holds.

Proof. Using a dyadic decomposition and Lemma 4.2 there exist $N_0 \geqslant 0$ such that if $N \geqslant N_0$ and, also, $m+\lambda+\frac{d}{p} > 0$ then

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus 2B}\frac{|f(y)|}{|x-y|^{d+m}}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-N}dy \\
&\leqslant C\sum_{k=1}^{\infty}(2^kR)^{-(d+m)}\Big(1+\frac{2^kR}{\rho(x)}\Big)^{-N}\int_{2^{k+1}B}f(y)\,dy \\
&\leqslant C\sum_{k=0}^{\infty}(2^kR)^{-(d+m)}\Big(1+\frac{2^kR}{\rho(x)}\Big)^{-(N-N_0)}2^{k(\frac{d}{p'}-\lambda)}\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\delta-\lambda}{d}}}\Big[\frac{f}{v}\Big]_p \\
&\leqslant C|B|^{-\frac{d+m}{d}}\sum_{k=0}^{\infty}2^{-k(m+\lambda+\frac{d}{p})}\frac{w(B)}{|B|^{\frac{1}{p}-\frac{\delta-\lambda}{d}}}\Big[\frac{f}{v}\Big]_p \\
&\leqslant C\frac{w(B)}{|B|^{1+\frac{1}{p}+\frac{m+\lambda-\delta}{d}}}\Big[\frac{f}{v}\Big]_p. \qquad \square
\end{aligned}$$

The next auxiliary lemma is a key tool to prove Theorem 1.1.

LEMMA 4.5. *1. Given s, ρ and β positive numbers and $N \geqslant 0$ there exists a constant C such that*

$$\int_0^\infty \frac{t^{-\frac{\beta}{2}}}{(1+\frac{\sqrt{t}}{\rho})^N} e^{-\frac{s^2}{t}} \frac{dt}{t} \leqslant C \frac{1}{s^\beta (1+\frac{s}{\rho})^N}.$$

2. Given $s > \sigma$, $\rho > 0$, β a real number, $N \geqslant 0$ and $M > \beta + N$ there is a constant C such that

$$\int_0^{\sigma^2} \frac{t^{-\frac{\beta}{2}}}{(1+\frac{\sqrt{t}}{\rho})^N} e^{-\frac{s^2}{t}} \frac{dt}{t} \leqslant C \Big(\frac{\sigma}{s}\Big)^{M-N-\beta} \frac{1}{s^\beta (1+\frac{s}{\rho})^N}.$$

Proof. (1) By the change of variable $u = \frac{s^2}{t}$ we get

$$\int_0^\infty \frac{t^{-\frac{\beta}{2}}}{(1+\frac{\sqrt{t}}{\rho})^N} e^{-\frac{s^2}{t}} \frac{dt}{t} = \frac{1}{s^\beta} \int_0^\infty \frac{u^{\frac{\beta}{2}}}{(1+\frac{s}{\rho\sqrt{u}})^N} e^{-u} \frac{du}{u}.$$

If $\frac{s}{\rho} < 1$ then $1 + \frac{s}{\rho} \leqslant 2$ and hence $\dfrac{u^{\frac{\beta}{2}}}{(1+\frac{s}{\rho\sqrt{u}})^N} \leqslant u^{\frac{\beta}{2}} \leqslant 2^N \dfrac{u^{\frac{\beta}{2}}}{(1+\frac{s}{\rho})^N}$. On the other hand, if $\frac{s}{\rho} > 1$ then $1 + \frac{s}{\rho} \leqslant 2\frac{s}{\rho}$ and thus $\dfrac{u^{\frac{\beta}{2}}}{(1+\frac{s}{\rho\sqrt{u}})^N} \leqslant \dfrac{u^{\frac{\beta+N}{2}}}{(\frac{s}{\rho})^N} \leqslant 2^N \dfrac{u^{\frac{\beta+N}{2}}}{(1+\frac{s}{\rho})^N}$. Therefore

$$\int_0^\infty \frac{t^{-\frac{\beta}{2}}}{(1+\frac{\sqrt{t}}{\rho})^N} e^{-\frac{s^2}{t}} \frac{dt}{t} \leqslant \frac{2^N}{s^\beta (1+\frac{s}{\rho})^N} \int_0^\infty (u^{\frac{\beta}{2}} + u^{\frac{\beta+N}{2}}) e^{-u} \frac{du}{u} \leqslant C_N \frac{1}{s^\beta (1+\frac{s}{\rho})^N}.$$

(2) By the same change of variable as above, the fact that $\sup_{t>1} t^{M/2} e^{-t} \leqslant C_M$ for any $M > 0$ and some $C_M > 0$, and following the steps in the above proof, it follows that

$$\begin{aligned}
\int_0^{\sigma^2} \frac{t^{-\frac{\beta}{2}}}{(1+\frac{\sqrt{t}}{\rho})^N} e^{-\frac{s^2}{t}} \frac{dt}{t} &= \frac{1}{s^\beta} \int_{(\frac{s}{\sigma})^2}^\infty \frac{u^{\frac{\beta}{2}}}{(1+\frac{s}{\rho\sqrt{u}})^N} e^{-u} \frac{du}{u} \\
&\leqslant C \frac{1}{s^\beta (1+\frac{s}{\rho})^N} \int_{(\frac{s}{\sigma})^2}^\infty (u^{\frac{\beta-M}{2}} + u^{\frac{\beta+N-M}{2}}) \frac{du}{u} \\
&\leqslant C \frac{1}{s^\beta (1+\frac{s}{\rho})^N} \Big((\frac{s}{\sigma})^{\beta-M} + (\frac{s}{\sigma})^{\beta+N-M} \Big) \\
&\leqslant C \frac{1}{s^\beta (1+\frac{s}{\rho})^N} \Big(\frac{s}{\sigma}\Big)^{\beta+N-M} \\
&\leqslant C \frac{\sigma^{M-N-\beta}}{s^{M-N} (1+\frac{s}{\rho})^N}. \qquad \square
\end{aligned}$$

4.1. Proof of Theorem 1.1

Let $(v,w) \in \mathscr{S}(q,\delta,\lambda)$ for some and $q<p$ or $v=w \in \mathscr{S}(p,\lambda,\lambda)$, i.e. $\delta=\lambda$ and split $f=f_1+f_2$, with $f_1=f\chi_{2B}$ and first assume that $R=\rho(x_B)$. Thus

$$\frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f(x)|dx \leqslant \frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f_1(x)|dx + \frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f_2(x)|dx \leqslant \mathscr{I}_1+\mathscr{I}_2.$$

By Lemma 4.3

$$\mathscr{I}_1 = \frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f_1(x)|dx \leqslant C|B|^{\frac{\alpha}{d}-\frac{1}{p}+\frac{\delta-\lambda}{d}}\left[\frac{f}{v}\right]_p. \tag{4.1}$$

To estimate $\mathscr{I}_2$ we display

$$|\mathscr{I}_\alpha f_2(x)| = \left|\int_0^\infty e^{-t\mathscr{L}} f_2(x)\, t^{\frac{\alpha}{2}}\,\frac{dt}{t}\right| \leqslant \int_0^\infty \int_{(2B)^c} k_t(x,y)\,|f(y)|\,dy\, t^{\frac{\alpha}{2}}\,\frac{dt}{t}.$$

Notice that if $x\in B$ and $y \in \mathbb{R}^d \setminus 2B$ then $2R \leqslant |x-y| \leqslant 2|x_B-y|$. Moreover, by Lemma 2.1,

$$\rho(x) \leqslant C\rho(x_B)\left(1+\frac{|x-x_B|}{\rho(x_B)}\right)^{\frac{k_0}{k_0+1}} \leqslant C\rho(x_B). \tag{4.2}$$

Hence, by Lemmas 2.1 and 4.5 given $N>0$ and $M>0$, to be chosen later, there is a constant C such that

$$\begin{aligned}
|\mathscr{I}_\alpha f_2(x)| &\leqslant C\int_{(2B)^c}\int_0^\infty t^{\frac{\alpha-d}{2}}\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^{-(N+M)} e^{-\frac{|x-y|^2}{t}}\,\frac{dt}{t}\,|f(y)|\,dy \\
&\leqslant C\int_{(2B)^c}\frac{|f(y)|}{|x-y|^{d-\alpha}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-(N+M)}dy \\
&\leqslant C\int_{(2B)^c}\frac{|f(y)|}{|x_B-y|^{d-\alpha}}\left(1+\frac{|x_B-y|}{\rho(x_B)}\right)^{-(N+M)}dy \\
&\leqslant C\rho(x_B)^M\int_{(2B)^c}\frac{|f(y)|}{|x_B-y|^{d-\alpha+M}}\left(1+\frac{|x_B-y|}{\rho(x_B)}\right)^{-N}dy.
\end{aligned}$$

By Lemma 4.4, taking $R=\rho(x_B)$, $N\geqslant N_0$ and choosing M such that $M-\alpha+\lambda+\frac{d}{p}>0$ we get

$$|\mathscr{I}_\alpha f_2(x)| \leqslant C\rho(x_B)^M \frac{w(B)}{|B|^{1+\frac{M-\alpha+\lambda-\delta}{d}+\frac{1}{p}}}\left[\frac{f}{v}\right]_p \leqslant C\frac{w(B)}{|B|^{1-\frac{\alpha+\delta-\lambda}{d}+\frac{1}{p}}}\left[\frac{f}{v}\right]_p.$$

Then

$$\mathscr{I}_2 = \frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f_2(x)|dx \leqslant C|B|^{\frac{\alpha}{d}-\frac{1}{p}+\frac{\delta-\lambda}{d}}\left[\frac{f}{v}\right]_p. \tag{4.3}$$

Estimates (4.1) and (4.3) give the proof of (1.9).

To prove (1.8), we consider $R<\rho(x_B)$, define $c_B=\int_{R^2}^\infty e^{-t\mathscr{L}} f_2(x_B) t^{\frac{\alpha}{2}}\frac{dt}{t}$ and split

$$\frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f(x)-c_B|dx \leqslant \frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f_1(x)|dx + \frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f_2(x)-c_B|dx.$$

As in (4.1) for $\alpha > 0$ it follows that

$$\frac{1}{w(B)}\int_B |\mathscr{I}_\alpha f_1(x)|dx \leqslant C|B|^{\frac{\alpha}{d}-\frac{1}{p}+\frac{\delta-\lambda}{d}}\left[\frac{f}{v}\right]_p.$$

On the other hand, for $x \in B$, we have

$$\begin{aligned}|\mathscr{I}_\alpha f_2(x) - c_B| &\leqslant \int_0^{R^2} |e^{-t\mathscr{L}} f_2(x)| t^{\frac{\alpha}{2}}\frac{dt}{t} + \int_{R^2}^{\infty} |e^{-t\mathscr{L}} f_2(x) - e^{-t\mathscr{L}} f_2(x_B)| t^{\frac{\alpha}{2}}\frac{dt}{t}\\ &= J_1(x) + J_2(x).\end{aligned}$$

If $|x-x_B| < R \leqslant \rho(x_B)$ then, as in (4.2), $\rho(x) \leqslant C\rho(x_B)$. Hence we apply Lemma 2.1, Lemma 4.5 with $\beta = d-\alpha$, $s = |x-y|$, $\sigma = R$, $M+d$ in place of M, and N positive such that $M > N+d-\alpha$, and use that if $|x-x_B| < R$ and $|y-x_B| \geqslant 2R$ then $|y-x| > R$, to get

$$\begin{aligned}J_1(x) &= \int_0^{R^2} |e^{-t\mathscr{L}} f_2(x)|\, t^{\frac{\alpha}{2}}\, \frac{dt}{t}\\ &\leqslant \int_0^{R^2} \left(\int_{(2B)^c} t^{-\frac{d}{2}}\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^{-N} e^{-\frac{|x-y|^2}{t}} |f(y)|dy\right) t^{\frac{\alpha}{2}}\, \frac{dt}{t}\\ &\leqslant C\int_{(2B)^c} \left(\int_0^{R^2} t^{\frac{\alpha-d}{2}}\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^{-N} e^{-\frac{|x-y|^2}{t}}\frac{dt}{t}\right) |f(y)|dy\\ &\leqslant CR^{\alpha+M-N}\int_{(2B)^c}\frac{|f(y)|}{|x-y|^{d+M-N}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} dy.\end{aligned}$$

By Lemma 4.4 if $M-N+\lambda+\frac{d}{p} > 0$ and $N \geqslant N_0$ for some N_0 big enough then

$$J_1(x) \leqslant C|B|^{\frac{\alpha}{d}+\frac{M-N}{d}}\frac{w(B)}{|B|^{1+\frac{M-N}{d}+\frac{1}{p}-\frac{\delta-\lambda}{d}}}\left[\frac{f}{v}\right]_p \leqslant C\frac{w(B)}{|B|^{1-\frac{\alpha+\delta-\lambda}{d}+\frac{1}{p}}}\left[\frac{f}{v}\right]_p \tag{4.4}$$

where we chose $M > \max(N+d-\alpha, N-\lambda-\frac{d}{p})$ and $N \geqslant N_0$.

On the other hand, we use Lemma 2.2 and Lemma 4.5 to get for any $\nu < \delta_0$ a constant C such that if $x \in B$ then

$$\begin{aligned}J_2(x) &= \int_{R^2}^{\infty} |e^{-t\mathscr{L}} f_2(x) - e^{-t\mathscr{L}} f_2(x_B)|\, t^{\frac{\alpha}{2}}\frac{dt}{t}\\ &= \int_{R^2}^{\infty}\int_{(2B)^c} |k_t(x,y) - k_t(x_B,y)|\, |f(y)|dy\, t^{\frac{\alpha}{2}}\frac{dt}{t}\\ &\leqslant C|x-x_B|^{\nu}\int_{R^2}^{\infty}\left(\int_{(2B)^c}\frac{t^{\frac{\alpha-d-\nu}{2}}}{(1+\frac{\sqrt{t}}{\rho(x)})^N}e^{-\frac{|x-y|^2}{t}}\, |f(y)|dy\right)\frac{dt}{t}\\ &\leqslant CR^{\nu}\int_{(2B)^c}\left(\int_{R^2}^{\infty}\frac{t^{\frac{\alpha-d-\nu}{2}}}{\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^N}e^{-\frac{|x-y|^2}{t}}\frac{dt}{t}\right)|f(y)|dy\\ &\leqslant CR^{\nu}\int_{(2B)^c}\frac{|f(y)|}{|x_B-y|^{d+\nu-\alpha}}\left(1+\frac{|x_B-y|}{\rho(x_B)}\right)^{-N} dy.\end{aligned}$$

Proceeding as in (4.4), since we can choose ν such that $\alpha-\lambda-\frac{d}{p}<\nu<\delta_0$ and apply Lemma 4.4, we get

$$J_2(x) \leqslant CR^{\nu}\frac{w(B)}{|B|^{\frac{d+\nu-\alpha-\delta+\lambda}{d}+\frac{1}{p}}}\left[\frac{f}{v}\right]_p = C\frac{w(B)}{|B|^{\frac{d-\alpha-\delta+\lambda}{d}+\frac{1}{p}}}\left[\frac{f}{v}\right]_p. \tag{4.5}$$

By integrating (4.4) and (4.5) on B we get (1.8).

The proof in the case $v=w\in\mathscr{S}(p,\lambda,\lambda)$ follows from Lemma 3.5, Remark 4.1 and the above reasoning. $\square$

5. Technical lemmas and proof of Theorem 1.2

In $BMO^{\beta}_{\mathscr{L}}(v)$ the average control is only on balls with radii greater than ρ at their centers (Corollary 2.1). However, for lower radii some kind of estimate can be proved. The following is a variation of Lemma 6 in [2].

LEMMA 5.1. *Let β,λ,δ and N_0 real numbers such that $\beta>\lambda$ and $\beta,N_0\geqslant 0$. If $(v,w)\in\mathscr{S}(\infty,\delta,\lambda,N_0)$ then there exist $C>0$ such that for any $f\in BMO^{\beta}_{\mathscr{L}}(v)$ and $k\in\mathbb{N}\cup\{0\}$*

$$\int_{2^kB}|f| \leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}|B|^{\frac{\beta-\lambda+\delta}{d}}w(B) \times\begin{cases}2^{k(d+\beta-\lambda+N_0)}\left(\frac{R}{\rho(x_B)}\right)^{N_0} & \text{if } k>j_0\\ 2^{kd}\left(\frac{\rho(x_B)}{R}\right)^{\beta-\lambda} & \text{if } k\leqslant j_0\end{cases} \tag{5.1}$$

for any ball $B=B(x_B,R)$ with $R\leqslant\rho(x_B)$ and $j_0\in\mathbb{N}\cup\{0\}$ such that $2^{j_0}R\leqslant\rho(x_B)<2^{j_0+1}R$.

Proof. Using Definitions 1.13 and 1.1 we consider two cases. If $k\geqslant j_0+1$, then for some $C>0$, $N_0\geqslant 0$ and any $f\in BMO^{\beta}_{\mathscr{L}}(v)$,

$$\begin{aligned}\int_{2^kB}|f| &\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}|2^kB|^{\frac{\beta}{d}}v(2^kB)\\ &\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}|2^kB|^{\frac{\beta-\lambda+d}{d}}\left(1+\frac{2^kR}{\rho(x_B)}\right)^{N_0}\frac{w(B)}{|B|^{1-\frac{\delta}{d}}}\\ &\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}2^{k(\beta-\lambda+d+N_0)}\left(\frac{R}{\rho(x_B)}\right)^{N_0}|B|^{\frac{\beta-\lambda+\delta}{d}}w(B).\end{aligned}$$

If $k\leqslant j_0$ and $\beta-\lambda>0$ by a dyadic decomposition and the previous inequality, then there exists a constant C such that for $f\in BMO^{\beta}_{\mathscr{L}}(v)$

$$\begin{aligned}
\frac{1}{|2^kB|}\int_{2^kB}|f| &\leqslant C\sum_{j=k}^{j_0+1}\frac{1}{|2^jB|}\int_{2^jB}|f(z)-f_{2^jB}|dz+\frac{1}{|2^{j_0+1}B|}\int_{2^{j_0+1}B}|f(z)|dz\\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\left\{\sum_{j=k}^{j_0+1}|2^jB|^{\frac{\beta}{d}}\frac{v(2^jB)}{|2^jB|}+2^{j_0(\beta-\lambda+N_0)}\left(\frac{R}{\rho(x_B)}\right)^{N_0}|B|^{\frac{\beta-\lambda+\delta}{d}}\frac{w(B)}{|B|}\right\}\\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\left\{\sum_{j=k}^{j_0+1}2^{j(\beta-\lambda)}\left(1+\frac{2^jR}{\rho(x_B)}\right)^{N_0}+\left(\frac{\rho(x_B)}{R}\right)^{\beta-\lambda}\right\}|B|^{\frac{\beta-\lambda+\delta}{d}}\frac{w(B)}{|B|}\\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\left\{\sum_{j=k}^{j_0+1}2^{j(\beta-\lambda)}+\left(\frac{\rho(x_B)}{R}\right)^{\beta-\lambda}\right\}|B|^{\frac{\beta-\lambda+\delta}{d}}\frac{w(B)}{|B|}\\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\left(\frac{\rho(x_B)}{R}\right)^{\beta-\lambda}|B|^{\frac{\beta-\lambda+\delta}{d}}\frac{w(B)}{|B|},
\end{aligned}$$

that is,

$$\int_{2^kB}|f|\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}2^{kd}\left(\frac{\rho(x_B)}{R}\right)^{\beta-\lambda}|B|^{\frac{\beta-\lambda+\delta}{d}}w(B). \quad \square$$

LEMMA 5.2. *Given $\beta\geqslant 0$, $\beta>\lambda$ with λ, δ and α real numbers and $N_0\geqslant 0$, let us assume that $(v,w)\in\mathscr{S}(\infty,\delta,\lambda,N_0)$.*

1. *If $M>N_0+\beta-\lambda+\alpha$ then for some constant $C=C_M$ and any ball $B=B(x_B,R)$ with $R\leqslant\rho(x_B)$,*

$$\int_{|x_B-y|\geqslant 2\rho(x_B)}\frac{|f(y)|}{|x_B-y|^{d+M-\alpha}}dy\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\frac{R^{\delta-d}}{\rho(x_B)^{-\beta+\lambda+M-\alpha}}\,w(B). \quad (5.2)$$

2. *If $M>\alpha$ then for some constant $C=C_M$ and any ball $B=B(x_B,R)$ with $R\leqslant\rho(x_B)$,*

$$\int_{2R\leqslant|x_B-y|<2\rho(x_B)}\frac{|f(y)|}{|x_B-y|^{d+M-\alpha}}dy\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\frac{R^{\delta-d-M+\alpha}}{\rho(x_B)^{-\beta+\lambda}}w(B) \quad (5.3)$$

for all $f\in BMO^{\beta}_{\mathscr{L}}(v)$.

Proof. Let $j_0\in\mathbb{N}\cup\{0\}$ such that $2^{j_0}R\leqslant\rho(x_B)<2^{j_0+1}R$. By a dyadic decomposing and the first item in Lemma 5.1, if $M>\beta-\lambda+\alpha+N_0$ then there exists a constant C such that

$$\begin{aligned}
&\int_{|x_B-y|\geqslant 2\rho(x_B)} \frac{|f(y)|}{|x_B-y|^{d+M-\alpha}}dy \leqslant \sum_{k=j_0}^{\infty} \frac{1}{(2^kR)^{d+M-\alpha}} \int_{2^{k+1}B} f(y)\,dy \\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \sum_{k=j_0}^{\infty} 2^{-k(-\beta-\alpha+\lambda+M-N_0)} \left(\frac{R}{\rho(x_B)}\right)^{N_0} R^{\beta-\lambda+\delta-d-M+\alpha} w(B) \\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} 2^{-j_0(-\beta-\alpha+\lambda+M-N_0)} \left(\frac{R}{\rho(x_B)}\right)^{N_0} R^{\beta-\lambda+\delta-d-M+\alpha} w(B) \\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \left(\frac{R}{\rho(x_B)}\right)^{-\beta-\alpha+\lambda+M} R^{\beta-\lambda+\delta-d-M+\alpha} w(B).
\end{aligned}$$

On the other hand, if $M>\alpha$ we get

$$\begin{aligned}
&\int_{2R\leqslant|x_B-y|<2\rho(x_B)} \frac{|f(y)|}{|x_B-y|^{d+M-\alpha}}dy \leqslant \sum_{k=0}^{j_0} \frac{1}{(2^kR)^{d+M-\alpha}} \int_{2^{k+1}B} f(y)\,dy \\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \sum_{k=0}^{j_0} \frac{1}{2^{k(M-\alpha)}} \left(\frac{\rho(x_B)}{R}\right)^{\beta-\lambda} R^{\beta-\lambda+\delta-d-M+\alpha} w(B) \\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \left(\frac{\rho(x_B)}{R}\right)^{\beta-\lambda} R^{\beta-\lambda+\delta-d-M+\alpha} w(B). \quad \square
\end{aligned}$$

In the following two lemmas we study certain kind of integrability for the oscillations of f.

LEMMA 5.3. *Let $\beta \geqslant 0$, λ and $\delta \in \mathbb{R}$, $\alpha \geqslant 0$ and $\beta > \lambda$.*

1. *If $(v,w) \in \mathscr{S}(\infty,\delta,\lambda,N_0)$, and $M > \beta+\alpha-\lambda+N_0$ then, for some $C>0$,*

$$\int_{|x_B-y|\geqslant 2\rho(x_B)} \frac{|f(y)-f_B|}{|x_B-y|^{d+M-\alpha}}dy \leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \frac{R^{\delta-d}}{\rho(x_B)^{-\beta+\lambda+M-\alpha}} w(B) \quad (5.4)$$

for any ball $B := B(x_B,R)$ with $R \leqslant \rho(x_B)$ and $f \in BMO^{\beta}_{\mathscr{L}}(v)$.

2. *If $(v,w) \in \mathscr{S}(\infty,\delta,\lambda)$, and $M > \beta-\lambda+\alpha$ then, for some $C>0$*

$$\int_{2R<|x_B-y|\leqslant 2\rho(x_B)} \frac{|f(y)-f_B|}{|y-x_B|^{d+M-\alpha}}dy \leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \frac{w(B)}{R^{-\beta+\lambda-\delta+M-\alpha+d}} \quad (5.5)$$

for any ball $B := B(x_B,R)$ with $R \leqslant \rho(x_B)$ and $f \in BMO^{\beta}_{\mathscr{L}}(v)$.

Proof. Let $j_0 \in \mathbb{N}\cup\{0\}$ such that $2^{j_0}R \leqslant \rho(x_B) < 2^{j_0+1}R$. Using a dyadic decomposition, Lemma 5.1 and Definition 1.1, and choosing M such that $M > N_0+\beta-$

$\lambda+\alpha>\alpha$ we get

$$\int_{|x_B-y|>2\rho(x_B)} \frac{|f(y)-f_B|}{|x_B-y|^{d+M-\alpha}}\,dy \leqslant C\sum_{j=j_0+1}^{\infty}\frac{1}{(2^jR)^{d+M-\alpha}}\int_{2^{j+1}B}|f(y)-f_B|\,dy$$

$$\leqslant C\sum_{j=j_0+1}^{\infty}\frac{1}{(2^jR)^{d+M-\alpha}}\Big\{\int_{2^{j+1}B}|f(y)|\,dy+|f_B|(2^jR)^d\Big\}$$

$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\frac{w(B)}{R^{d+M-\alpha-\beta-\delta+\lambda}}$$

$$\times\sum_{j=j_0+1}^{\infty}2^{-j(d+M-\alpha)}\Big\{2^{j(d+\beta-\lambda+N_0)}\Big(\frac{R}{\rho(x_B)}\Big)^{N_0}+2^{jd}\Big(\frac{\rho(x_B)}{R}\Big)^{\beta-\lambda}\Big\}$$

$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\frac{w(B)}{R^{d+M-\alpha-\beta-\delta+\lambda}}$$

$$\times\Big\{\Big(\frac{R}{\rho(x_B)}\Big)^{N_0}\sum_{j=j_0+1}^{\infty}2^{j(\beta-\lambda-M+\alpha+N_0)}+\Big(\frac{\rho(x_B)}{R}\Big)^{\beta-\lambda}\sum_{j=j_0+1}^{\infty}2^{-j(M-\alpha)}\Big\}$$

$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\Big(\frac{R}{\rho(x_B)}\Big)^{-\beta+\lambda+M-\alpha}\frac{w(B)}{R^{d+M-\alpha-\beta-\delta+\lambda}}.$$

On the other hand, by Definition 1.1 and (2.2) if $M>\beta-\lambda+\alpha>\alpha$ it follows that for some $C>0$,

$$\int_{2R<|x_B-y|\leqslant 2\rho(x_B)}\frac{|f(y)-f_B|}{|x_B-y|^{d+M-\alpha}}\,dy\leqslant C\sum_{j=1}^{j_0}\frac{1}{(2^jR)^{d+M-\alpha}}\int_{2^{j+1}B}|f(y)-f_B|\,dy$$

$$\leqslant\sum_{j=1}^{j_0}\frac{1}{(2^jR)^{M-\alpha}}\Big\{\frac{1}{(2^{j+1}R)^d}\int_{2^{j+1}B}|f(y)-f_{2^{j+1}B}|\,dy+\sum_{k=0}^{j}|f_{2^{k+1}B}-f_{2^kB}|\Big\}$$

$$\leqslant C\sum_{j=1}^{j_0}\frac{1}{(2^jR)^{M-\alpha}}\sum_{k=0}^{j+1}\frac{1}{(2^kR)^d}\int_{2^kB}|f(y)-f_{2^kB}|\,dy$$

$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\sum_{j=1}^{j_0}\frac{1}{(2^jR)^{M-\alpha}}\sum_{k=1}^{j+1}(2^kR)^{\beta}\frac{v(2^kB)}{(2^kR)^d}$$

$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\sum_{j=1}^{j_0}\frac{1}{2^{j(M-\alpha)}}\sum_{k=1}^{j+1}2^{k\beta}2^{-k\lambda}\Big(1+\frac{2^kR}{\rho(x_B)}\Big)^{N_0}\frac{w(B)}{R^{d-\beta+\lambda-\delta+M-\alpha}}$$

$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\sum_{k=1}^{j_0+1}2^{k(\beta-\lambda)}\sum_{j=k-1}^{j_0+1}\frac{1}{2^{j(M-\alpha)}}\frac{w(B)}{R^{d-\beta+\lambda-\delta+M-\alpha}}$$

$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\sum_{k=1}^{j_0+1}2^{k(\beta-\lambda-M+\alpha)}\frac{w(B)}{R^{d-\beta+\lambda-\delta+M-\alpha}}$$

$$\leqslant\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\frac{w(B)}{R^{d-\beta+\lambda-\delta+M-\alpha}}.\quad\square$$

Since by Proposition 1.1 the pointwise regularity conditions (1.17) and (2.3) characterize a Campanato-type space then the proof of Theorem 1.2 will follow from the verification of those two conditions and using Remark 1.1.

LEMMA 5.4. *Let $\beta \geqslant 0$, $\beta > \lambda$ and $\delta \in \mathbb{R}$. If $(v,w) \in \mathscr{S}(\infty,\delta,\lambda)$ and $\alpha > 0$ then there exists a constant C such that for all $f \in BMO^{\beta}_{\mathscr{L}}(v)$,*

$$\frac{1}{w(B(x_B,\rho(x_B)))}\int_{B(x_B,\rho(x_B))}|\mathscr{I}_\alpha f(x)|dx \leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}\rho(x_B)^{\beta-\lambda+\delta+\alpha} \tag{5.6}$$

for any $x_B \in \mathbb{R}^d$.

Proof. Let us denote $\rho := \rho(x_B), B := B(x_B,\rho)$ and split, $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. Hence by inequality (2.8), Lemma 5.1 and the fact that $\alpha > 0$ it follows that

$$\begin{aligned}\frac{1}{w(B)}\int_B|\mathscr{I}_\alpha f_1(x)|dx &\leqslant \frac{1}{w(B)}\int_B\int_{2B}\frac{|f(y)|}{|x-y|^{d-\alpha}}dydx \leqslant C\rho^\alpha\frac{1}{w(B)}\int_{2B}|f(y)|dy\\ &\leqslant C\rho^{\beta-\lambda+\delta+\alpha}\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}.\end{aligned} \tag{5.7}$$

On the other hand, if $x \in B$ and $y \in (2B)^c$ then $|x-y| \geqslant |x_B-y|/2 \geqslant \rho$ and, by (4.2), $\rho(x) \leqslant C\rho$. Hence, by Lemma 2.1, given $N > 0$ there is a constant C such that

$$|k_t(x,y)| \leqslant C\frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}}\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^N} \leqslant C\rho^N\frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{d+N}{2}}}. \tag{5.8}$$

Thus, if $N > \alpha - d$, using the change of variable $s = \frac{|x-y|^2}{t}$, we get

$$\begin{aligned}|\mathscr{I}_\alpha f_2(x)| &\leqslant \int_0^\infty t^{\frac{\alpha}{2}}\Big(\int_{(2B)^c}|k_t(x,y)|\,|f(y)|\,dy\Big)\,\frac{dt}{t}\\ &\leqslant C\rho^N\int_{(2B)^c}|f(y)|\Big(\int_0^\infty\frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{d+N-\alpha}{2}}}\,\frac{dt}{t}\Big)\,dy\\ &\leqslant C\rho^N\int_0^\infty s^{\frac{d+N-\alpha}{2}}e^{-s}\frac{ds}{s}\int_{(2B)^c}\frac{|f(y)|}{|x-y|^{d+N-\alpha}}dy\\ &\leqslant C\rho^N\int_{(2B)^c}\frac{|f(y)|}{|x_B-y|^{d+N-\alpha}}dy.\end{aligned}$$

Since $(v,w) \in \mathscr{S}(\infty,\delta,\lambda,N_0)$ for some $N_0 \geqslant 0$, using (5.2) and taking $N > N_0+\beta-\lambda+\alpha$, we obtain

$$|\mathscr{I}_\alpha f_2(x)| \leqslant C\rho^N\frac{w(B)}{\rho^{d-\alpha-\beta-\delta+\lambda+N}}\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \leqslant C\rho^{\beta-\lambda+\alpha+\delta}\frac{w(B)}{\rho^d}\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}.$$

Hence

$$\frac{1}{w(B)}\int_B|\mathscr{I}_\alpha f_2(x)|dx \leqslant C\rho^{\beta-\lambda+\alpha+\delta}\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)}. \tag{5.9}$$

With (5.7) and (5.9) we get the proof of the lemma. □

LEMMA 5.5. *Let $\beta \geqslant 0$, $\alpha > 0$, $\lambda < \beta$, $\alpha+\beta-\lambda < \delta_0$ and $\alpha+\beta-\lambda+\delta \leqslant d$. If w is a doubling weight and $(v,w) \in \mathscr{S}(\infty,\delta,\lambda)$, then there exists a constant C such that*

$$|\mathscr{I}_\alpha f(x) - \mathscr{I}_\alpha f(y)| \leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(v)}(W_{\beta-\lambda+\alpha+\delta}(x,|x-y|) + W_{\beta-\lambda+\alpha+\delta}(y,|x-y|))$$

for $|x-y| < \rho(x)$, $f \in BMO^\beta_{\mathscr{L}}(v)$ and W defined as in (1.14).

Proof. Set $R = |x-y| < \rho(x)$ and denote $B = B(x,R)$. Then

$$\begin{aligned}|\mathscr{I}_\alpha f(x) - \mathscr{I}_\alpha f(y)| &\leqslant \int_{\mathbb{R}^d}\int_0^\infty |k_t(x,z)-k_t(y,z)|t^{\frac{\alpha}{2}}\frac{dt}{t}|f(z)|\,dz \\ &\leqslant \Big(\int_0^{\rho(x)^2} + \int_{\rho(x)^2}^\infty\Big)t^{\frac{\alpha}{2}}\int_{\mathbb{R}^d}|k_t(x,z)-k_t(y,z)|\,|f(z)|\,dz\,\frac{dt}{t} = I + J.\end{aligned}$$

By splitting J we get

$$J = \int_{\rho(x)^2}^\infty t^{\frac{\alpha}{2}}\Big(\int_{|x-z|\leqslant 4\rho(x)} + \int_{|x-z|>4\rho(x)}\Big)|k_t(x,z)-k_t(y,z)|\,|f(z)|\,dz\,\frac{dt}{t} = J_1 + J_2.$$

Let $j_0 \in \mathbb{N}\cup\{0\}$ such that $2^{j_0}R \leqslant \rho(x) < 2^{j_0+1}R$. Since $\sqrt{t} > \rho(x) > R = |x-y|$ then by Lemma 2.2 and arguments similar to those used to obtain (5.8), given $N > 0$, $\nu < \delta_0$ and $L > 0$ there exists a constant $C = C_{\nu,N,L}$ such that

$$\begin{aligned}|k_t(x,z)-k_t(y,z)| &\leqslant C\Big(\frac{R}{\sqrt{t}}\Big)^\nu t^{-\frac{d}{2}}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-N}e^{-\frac{|x-z|^2}{t}} \\ &\leqslant CR^\nu\rho(x)^N \times \begin{cases} t^{-\frac{d+\nu+N}{2}} & \text{if } \frac{|x-z|^2}{t} \leqslant 16 \\ \frac{t^{-\frac{d+\nu+N-L}{2}}}{|x-z|^L} & \text{if } \frac{|x-z|^2}{t} > 16.\end{cases}\end{aligned} \tag{5.10}$$

In the case $\frac{|x-z|^2}{t} \leqslant 16$ choosing ν such that $\alpha+\beta-\lambda \leqslant \nu < \delta_0$ and $N > 0$ such that $N > d-\alpha+\nu$, using (5.10) and Lemma 5.1, since $\alpha+\beta-\lambda+\delta \leqslant d$ we can obtain

$$\begin{aligned}J_1 &\leqslant CR^\nu\rho(x)^N\int_{\rho(x)^2}^\infty t^{\frac{\alpha-d-\nu-N}{2}}\frac{dt}{t}\int_{|x-z|<2^{j_0+3}R}|f(z)|dz \\ &\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(v)}R^\nu\rho(x)^{\alpha-d-\nu}2^{j_0(d+\beta-\lambda+N_0)}\Big(\frac{R}{\rho(x)}\Big)^{N_0}R^{\beta-\lambda+\delta}w(B) \\ &\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(v)}\Big(\frac{R}{\rho(x)}\Big)^{-\beta+\lambda+\nu-\alpha}R^{\beta-\lambda+\alpha+\delta-d}w(B) \\ &\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(v)}R^{\beta-\lambda+\alpha+\delta-d}w(B) \leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(v)}W_{\beta-\lambda+\alpha+\delta}(x,R)\end{aligned}$$

for some constants $C > 0$ and $N_0 \geqslant 0$.

In the case $\frac{|x-z|^2}{t} > 16$ setting $N_0 \geqslant 0$ such that $(v,w) \in \mathscr{S}(\infty,\delta,\lambda,N_0)$, ν such that $\beta-\lambda+\alpha < \nu$, choosing $M_1 > N_0+\beta-\lambda+\alpha-\nu$, defining $M = M_1+\nu$ in

(5.2), choosing $N > M_1 + \nu$ and $L = d - \alpha + M_1 + \nu$ in (5.10) and finally assuming that $\beta - \lambda + \alpha + \delta \leqslant d$ there exists a constant C such that

$$\begin{aligned}
J_2 &\leqslant C\, R^\nu \rho(x)^N \int_{\rho(x)^2}^{\infty} t^{\frac{\alpha-d-\nu-N}{2}} t^{\frac{d-\alpha+\nu+M_1}{2}} \frac{dt}{t} \int_{|x-z|\geqslant 4\rho(x)} \frac{|f(z)|}{|x-z|^{d-\alpha+\nu+M_1}} dz \\
&\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(\nu)} R^\nu \rho(x)^N \int_{\rho(x)^2}^{\infty} t^{\frac{-N+M_1}{2}} \frac{dt}{t} \Big(\frac{R}{\rho(x)}\Big)^{-\beta+\lambda-\alpha+\nu+M_1} \frac{R^{\beta-\lambda+\delta} w(B)}{R^{d-\alpha+\nu+M_1}} \\
&\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(\nu)} \Big(\frac{R}{\rho(x)}\Big)^{-\beta+\lambda+\nu-\alpha} R^{\beta-\lambda+\alpha+\delta-d} w(B) \\
&\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(\nu)} W_{\beta-\lambda+\alpha+\delta}(x,R).
\end{aligned}$$

To deal with I we split

$$I \leqslant \int_0^{\rho(x)^2} t^{\frac{\alpha}{2}} \int_{\mathbb{R}^d} \Big(|q_t(x,z) - q_t(y,z)| + |\tilde{k}_t(x,z) - \tilde{k}_t(y,z)|\Big) |f(z)|\, dz \frac{dt}{t} = I_1 + I_2,$$

where $\tilde{k}$ is the kernel of the classical heat operator $e^{-t\Delta}$ and $q_t = k_t - \tilde{k}_t$. For the first term above we get that

$$\begin{aligned}
I_1 \leqslant \int_0^{\rho(x)^2} t^{\frac{\alpha}{2}} \int_{\mathbb{R}^d} \Big(&\chi_{\{|x-z|>4R\}} |q_t(x,z) - q_t(y,z)| \\
&+ \chi_{\{|x-z|\leqslant 4R\}} |q_t(x,z)| + \chi_{\{|x-z|\leqslant 4R\}} |q_t(y,z)|\Big) |f(z)|\, dz \frac{dt}{t} \\
=& I_{1,1} + I_{1,2} + I_{1,3}.
\end{aligned}$$

By Lemma 2.4, in the case $t < \rho(x)^2$ and any $N > 0$ there exists C (independent of t)

$$|q_t(x,z) - q_t(y,z)| \leqslant C\Big(\frac{R}{\rho(x)}\Big)^\nu \frac{1}{t^{\frac{d}{2}}(1 + \frac{|x-z|}{\sqrt{t}})^{d+N}} \leqslant C\Big(\frac{R}{\rho(x)}\Big)^\nu \frac{t^{\frac{N}{2}}}{|x-z|^{d+N}},$$

where $R = |x-y| < C\rho(x)$ and $|x-z| > 4R$.

To estimate $I_{1,1}$ we split $\{z : |x-z| > 4R\} = \{z : 4R < |x-z| \leqslant 4\rho(x)\} \cup \{z : |x-z| > 4\rho(x)\}$, define in the above inequality $N = \xi$ with $0 < \xi < \nu - (\alpha + \beta - \lambda)$ for the first domain and $N = N_2 > N_0 + \beta - \lambda$ for the second and use Lemma 5.2 and condition $\alpha + \beta - \lambda + \delta \leqslant d$ to get, for some constant C,

$$\begin{aligned}
I_{1,1} \leqslant C\Big(\frac{R}{\rho(x)}\Big)^\nu \Big(&\int_0^{\rho(x)^2} t^{\frac{\alpha+\xi}{2}} \frac{dt}{t} \int_{\{4R<|x-z|\leqslant 4\rho(x)\}} \frac{|f(z)|}{|x-z|^{d+\xi}} dz \\
&+ \int_0^{\rho(x)^2} t^{\frac{\alpha+N_2}{2}} \frac{dt}{t} \int_{\{|x-z|>4\rho(x)\}} \frac{|f(z)|}{|x-z|^{d+N_2}} dz\Big) \\
\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(\nu)} &\Big(\Big(\frac{\rho(x)}{R}\Big)^{\xi+\beta-\lambda+\alpha-\nu} + \Big(\frac{R}{\rho(x)}\Big)^{-\beta+\lambda+\nu-\alpha}\Big) \frac{R^{\beta-\lambda+\alpha+\delta} w(B)}{R^d} \\
\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(\nu)} &R^{\beta-\lambda+\alpha+\delta-d} w(B) \\
\leqslant C\|f\|_{BMO^\beta_{\mathscr{L}}(\nu)} &W_{\beta-\lambda+\alpha+\delta}(x,R). \qquad (5.11)
\end{aligned}$$

On the other hand, by Lemma 2.3, for $\sqrt{t}<\rho$ we get

$$|q_t(x,z)| \leqslant C_M \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu} t^{-\frac{d}{2}} \Big(1+\frac{|x-z|}{\sqrt{t}}\Big)^{-M}.$$

If $\sqrt{t}<R/4$ then we set an integer $n_t \geqslant 2$ such that $2^{n_t}\sqrt{t} \leqslant R < 2^{n_t+1}\sqrt{t}$, use a dyadic decomposition and the above inequality to get

$$\begin{aligned} I_{1,2} \leqslant C\Big(&\int_0^{\frac{R^2}{16}} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu} t^{\frac{\alpha-d}{2}} \\ &\times \Big(\int_{|x-z|<4\sqrt{t}} |f(z)|dz + \sum_{j=2}^{n_t} \int_{2^j\sqrt{t}\leqslant |x-z|<2^{j+1}\sqrt{t}} \Big(\frac{\sqrt{t}}{|x-z|}\Big)^{M} |f(z)|dz\Big)\frac{dt}{t} \\ &+ \int_{\frac{R^2}{16}}^{\frac{\rho(x)^2}{16}} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu} t^{\frac{\alpha-d}{2}} \frac{dt}{t} \int_{|x-z|\leqslant 2R} |f(z)|\,dz\Big) = H_1 + H_2. \end{aligned}$$

Now, since $\beta-\lambda>0$ then we are able to apply Lemma 5.1, that w is doubling, $\alpha+\beta-\lambda+\delta \leqslant d$, $\beta-\lambda<\beta-\lambda+\alpha<\nu$ and choose $M>d$ to obtain

$$\begin{aligned} H_1 \leqslant & C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \int_0^{\frac{R^2}{16}} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu} t^{\frac{\alpha-d}{2}} \\ &\times \Big(4^d \Big(\frac{\rho(x)}{\sqrt{t}}\Big)^{\beta-\lambda} t^{\frac{\beta-\lambda+\delta}{2}} w(B(x,4\sqrt{t})) + \sum_{j=2}^{n_t} 2^{-jM} \int_{|x-z|<2^{j+1}\sqrt{t}} |f(z)|dz\Big)\frac{dt}{t} \\ \leqslant & C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \int_0^{\frac{R^2}{16}} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu} t^{\frac{\beta-\lambda+\alpha+\delta-d}{2}} \\ &\times \Big(\frac{\rho(x)}{\sqrt{t}}\Big)^{\beta-\lambda} \Big(1+\sum_{j=2}^{n_t} 2^{-jM+jd}\Big) w(B(x,4\sqrt{t}))\frac{dt}{t} \\ \leqslant & C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \int_0^{\frac{R^2}{16}} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu-\beta+\lambda} \frac{dt}{t}\, W_{\beta-\lambda+\alpha+\delta}(x,R) \\ \leqslant & C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} W_{\beta-\lambda+\alpha+\delta}(x,R). \end{aligned}$$

Analogously, since $\beta-\lambda+\alpha+\delta \leqslant d$ and $\beta-\lambda<\beta-\lambda+\alpha<\nu\leqslant d$ then

$$\begin{aligned} H_2 \leqslant & C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \int_{\frac{R^2}{16}}^{\frac{\rho(x)^2}{16}} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu} t^{\frac{\alpha-d}{2}} \frac{dt}{t}\, 2^d \Big(\frac{\rho(x)}{R}\Big)^{\beta-\lambda} R^{\delta+\beta-\lambda} w(B(x,4R)) \\ \leqslant & C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \int_{\frac{R^2}{16}}^{\frac{\rho(x)^2}{16}} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu+\alpha-d} \frac{dt}{t} \Big(\frac{\rho(x)}{R}\Big)^{\beta-\lambda+\alpha-d} R^{\beta-\lambda+\delta+\alpha-d} w(B(x,R)) \\ \leqslant & C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \int_{\frac{R^2}{16}}^{\frac{\rho(x)^2}{16}} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\beta-\lambda+\alpha-d} \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\nu-\beta+\lambda} \frac{dt}{t} \\ &\times \Big(\frac{\rho(x)}{R}\Big)^{\beta-\lambda+\alpha-d} W_{\beta-\lambda+\alpha+\delta}(x,R) \end{aligned}$$

$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \int_0^{\frac{\rho(x)^2}{16}} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{v-\beta+\lambda} \frac{dt}{t}\ W_{\beta-\lambda+\alpha+\delta}(x,R)$$
$$\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} W_{\beta-\lambda+\alpha+\delta}(x,R).$$

Hence

$$I_{1,2} \leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} W_{\beta-\lambda+\alpha+\delta}(x,R). \tag{5.12}$$

Since $I_{1,3}$ is similar to $I_{1,2}$ we also obtain

$$I_{1,3} \leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} W_{\beta-\lambda+\alpha+\delta}(x,R). \tag{5.13}$$

Inequalities (5.11), (5.12) and (5.13) give the seek estimate of I_1.

On the other hand, since $\int_{\mathbb{R}^d} (\tilde{k}_t(x,z) - \tilde{k}_t(y,z))\,dz = 0$ then

$$\begin{aligned}
I_2 &\leqslant \int_0^{\rho(x)^2} t^{\frac{\alpha}{2}} \int_{\mathbb{R}^d} |\tilde{k}_t(x,z) - \tilde{k}_t(y,z)|\,|f(z) - f_B|\,dz \frac{dt}{t} \\
&\leqslant \int_0^{\rho(x)^2} t^{\frac{\alpha}{2}} \Big(\int_{|x-z|>2R} |\tilde{k}_t(x,z) - \tilde{k}_t(y,z)|\,|f(z) - f_B|\,dz \\
&\quad + \int_{|x-z|\leqslant 2R} |\tilde{k}_t(x,z)|\,|f(z) - f_B|\,dz + \int_{|x-z|\leqslant 2R} |\tilde{k}_t(y,z)|\,|f(z) - f_B|\,dz \Big) \frac{dt}{t} \\
&= I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned}$$

It is not difficult to see that $|\tilde{k}_t(x,z) - \tilde{k}_t(y,z)| \leqslant C e^{-\frac{|x-z|^2}{ct}} t^{-\frac{d}{2}-1}|x-y||x-z|$ for $|x-z| > 2|x-y|$ and $\alpha < \beta - \lambda + \alpha < \delta_0 \leqslant 1$ then, choosing $M > N_0 - \lambda + \alpha + \beta - 1$ and using (5.5) and (5.4) in Lemma 5.3, it follows that

$$\begin{aligned}
I_{2,1} &\leqslant CR \int_{|x-z|>2R} |x-z||f(z) - f_B| \int_0^{\rho(x)^2} t^{\frac{\alpha-d-2}{2}} e^{-\frac{|x-z|^2}{ct}} \frac{dt}{t}\,dz \\
&\leqslant CR \int_{|x-z|>2R} \frac{|f(z) - f_B|}{|x-z|^{d+1-\alpha}} \int_{\frac{|x-z|^2}{c\rho(x)^2}}^{\infty} s^{\frac{d+2-\alpha}{2}} e^{-s} \frac{ds}{s}\,dz \\
&\leqslant CR \Big(\int_{2R<|x-z|\leqslant 2\rho(x)} \frac{|f(z) - f_B|}{|x-z|^{d+1-\alpha}}\,dz \Big(\int_0^{\infty} s^{\frac{d+2-\alpha}{2}} e^{-s} \frac{ds}{s} \Big) \\
&\qquad + \int_{|x-z|>2\rho(x)} \frac{|f(z) - f_B|}{|x-z|^{d+1-\alpha}} \int_{\frac{|x-z|^2}{c\rho(x)^2}}^{\infty} s^{\frac{d+2-\alpha-M}{2}} e^{-\frac{s}{c}} \frac{ds}{s}\,dz \Big) \\
&\leqslant CR \Big(\int_{2R<|x-z|\leqslant 2\rho(x)} \frac{|f(z) - f_B|}{|x-z|^{d+1-\alpha}}\,dz + \rho(x)^M \int_{|x-z|>2\rho(x)} \frac{|f(z) - f_B|}{|x-z|^{d-\alpha+1+M}}\,dz \Big) \\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \frac{w(B)}{R^{d-\beta-\delta+\lambda-\alpha}} \leqslant C\|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} W_{\beta-\lambda+\alpha+\delta}(x,R).
\end{aligned}$$

For $I_{2,2}$ we use that $(v,w) \in \mathscr{S}(\infty,\delta,\lambda)$ and that w is doubling to get

$$
\begin{aligned}
I_{2,2} &\leqslant C \int_{|x-z|\leqslant 2R} |f(z)-f_B| \int_0^{\rho(x)^2} t^{\frac{\alpha-d}{2}} e^{-\frac{|x-z|^2}{ct}} \frac{dt}{t}\, dz \\
&\leqslant C \int_{|x-z|\leqslant 2R} \frac{|f(z)-f_B|}{|x-z|^{d-\alpha}} \Big(\int_{\frac{|x-z|^2}{c\rho(x)^2}}^{\infty} s^{\frac{-\alpha+d}{2}} e^{-s} \frac{ds}{s} \Big)\, dz \\
&\leqslant C \int_{|x-z|\leqslant 2R} \frac{|f(z)-f_B|}{|x-z|^{d-\alpha}}\, dz \leqslant C \sum_{j=0}^{\infty} (2^{-j}R)^{\alpha-d} \int_{|x-z|<2^{-j+1}R} |f(z)-f_B|\, dz \\
&\leqslant C \|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \sum_{j=0}^{\infty} (2^{-j}R)^{\beta+\alpha-d} v(2^{-j+1}B) \\
&\leqslant C \|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \sum_{j=0}^{\infty} (2^{-j}R)^{\beta-\lambda+\alpha+\delta-d} w(2^{-j+1}B) \\
&\leqslant C \|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \sum_{j=1}^{\infty} (2^{-j}R)^{\beta-\lambda+\alpha+\delta-d} w(2^{-j+1}B \setminus 2^{-j}B) \\
&\leqslant C \|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} \int_{|x-z|\leqslant 2R} \frac{w(z)}{|x-z|^{d-\beta+\lambda-\alpha-\delta}}\, dz = C \|f\|_{BMO^{\beta}_{\mathscr{L}}(v)} W_{\beta-\lambda+\alpha+\delta}(x,R).
\end{aligned}
$$

The estimate for $I_{2,3}$ is obtained in the same way. □

5.1. Proof of Theorem 1.2

Since $\beta-\lambda+\alpha+\delta<1$ and w is doubling then to prove this Theorem it will be enough to use the pointwise characterization of the space $BMO^{\alpha+\beta-\lambda+\delta}_{\mathscr{L}}(w)$ provided by Proposition 1.1. But then Lemma 5.4 and Lemma 5.5 give the two desired inequalities for $\mathscr{I}_\alpha f$.

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