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EXACT LOWER AND UPPER BOUNDS ON THE INCOMPLETE GAMMA FUNCTION

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Abstract. Lower and upper bounds $B_a(x)$ on the incomplete gamma function $\Gamma(a,x)$ are given for all real a and all real $x>0$. These bounds $B_a(x)$ are exact in the sense that $B_a(x) \underset{x\downarrow 0}{\sim} \Gamma(a,x)$ and $B_a(x) \underset{x\to\infty}{\sim} \Gamma(a,x)$. Moreover, the relative errors of these bounds are rather small for other values of x, away from 0 and ∞.

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REFERENCES

[1] MATHEMATICAL REVIEWS, *Review MR0103289*, 1960., https://mathscinet.ams.org/mathscinet-getitem?mr=MR0103289.
[2] H. ALZER, *On some inequalities for the incomplete gamma function*, Math. Comp., **66** (218): 771–778, 1997.
[3] H. ALZER AND A. BARICZ, *Functional inequalities for the incomplete gamma function*, J. Math. Anal. Appl., **385** (1): 167–178, 2012.
[4] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special functions, volume 71 of Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1999.
[5] J. M. BORWEIN AND O.-Y. CHAN, *Uniform bounds for the complementary incomplete gamma function*, Math. Inequal. Appl., **12** (1): 115–121, 2009.
[6] W. GAUTSCHI, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. and Phys., **38**: 77–81, 1959/60.
[7] W. GAUTSCHI, *Personal communication*, 2019.
[8] P. GREENGARD AND V. ROKHLIN, *An algorithm for the evaluation of the incomplete gamma function*, Adv. Comput. Math., **45** (1): 23–49, 2019.
[9] A. LAFORGIA AND P. NATALINI, *Supplements to known monotonicity results and inequalities for the gamma and incomplete gamma functions*, J. Inequal. Appl., pages Art. ID 48727, 8, 2006.
[10] P. NATALINI AND B. PALUMBO, *Inequalities for the incomplete gamma function*, Math. Inequal. Appl., **3** (1): 69–77, 2000.
[11] E. NEUMAN, *Inequalities and bounds for the incomplete gamma function*, Results Math., **63** (3–4): 1209–1214, 2013.
[12] R. B. PARIS, *Error bounds for the uniform asymptotic expansion of the incomplete gamma function*, J. Comput. Appl. Math., **147** (1): 215–231, 2002.
[13] I. PINELIS, *L'Hospital type rules for oscillation, with applications*, JIPAM. J. Inequal. Pure Appl. Math., **2** (3): Article 33, 24 pp. (electronic), 2001.
[14] I. PINELIS, *On l'Hospital-type rules for monotonicity*, JIPAM – J. Inequal. Pure Appl. Math., **7** (2): Article 40, 19 pp. (electronic), www.emis.de/journals/JIPAM/images/157_05_JIPAM/157_05.pdf, 2006.
[15] F. QI, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl., **5** (1): 61–67, 2002.
[16] F. QI AND S.-L. GUO, *Inequalities for the incomplete gamma and related functions*, Math. Inequal. Appl., **2** (1): 47–53, 1999.

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[17] N. M. TEMME, *The asymptotic expansion of the incomplete gamma functions*, SIAM J. Math. Anal., **10** (4): 757–766, 1979.
[18] Z.-H. YANG, W. ZHANG, AND Y.-M. CHU, *Sharp Gautschi inequality for parameter* $0 < p < 1$ *with applications*, Math. Inequal. Appl., **20** (4): 1107–1120, 2017.