

## **$n$ -DERIVATIONS AND FUNCTIONAL INEQUALITIES WITH APPLICATIONS**

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*Abstract.* We prove that every bounded  $n$ -derivation of a commutative factorizable Banach algebra maps into its radical. Also, the nilpotency of eigenvectors of any bounded  $n$ -derivation corresponding to its eigenvalues is derived. We introduce the notion of approximate  $n$ -derivations on a Banach algebra  $\mathcal{A}$  and show that the separating space of an approximate  $n$ -derivation ( $n > 2$ ) is not necessarily an ideal, unless the Banach algebra  $\mathcal{A}$  is factorizable. From this and some results on bounded  $n$ -derivations, we prove that every approximate  $n$ -derivation of a semisimple factorizable Banach algebra is automatically continuous and every approximate  $n$ -derivation of a commutative semisimple factorizable Banach algebra is identically zero. Some applications of our results are also provided.

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