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SHARP INEQUALITIES FOR THE NUMERICAL RADIUS OF HILBERT SPACE OPERATORS AND OPERATOR MATRICES

PINTU BHUNIA, KALLOL PAUL AND RAJ KUMAR NAYAK

Abstract. We present new upper and lower bounds for the numerical radius of a bounded linear operator defined on a complex Hilbert space, which improve on the existing bounds. Among many other inequalities proved in this article, we show that for a non-zero bounded linear operator T on a complex Hilbert space H,

$$w(T) \geqslant \frac{\|T\|}{2} + \frac{m(T^2)}{2\|T\|},$$

where $w(T)$ is the numerical radius of T and $m(T^2)$ is the Crawford number of T^2. This substantially improves on the existing inequality $w(T) \geqslant \frac{\|T\|}{2}$. We also obtain some upper and lower bounds for the numerical radius of operator matrices and illustrate with numerical examples that these bounds are better than the existing bounds.

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