

MEANS PRODUCED BY DISTANCES

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Abstract. We describe a methodology that can be used to construct new distances which produce many famous means. Its main application is to construct a distance for the logarithmic mean, settling an old open problem. We also use it to construct alternative distances for already known means, such as the arithmetic and all quasi-arithmetic means. Moreover, we show how to construct distances for almost all means that can be obtained from Cauchy's Mean Value Theorem, and apply this to construct distances for all Stolarsky means. Finally, we show how to construct a distance for a mean $\mathcal{M}_q(a, b) = q^{-1}(\mathcal{M}(q(a), q(b)))$, where \mathcal{M} is another mean for which a distance is already known, and q is a monotone bijection to a subinterval.

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