INEQUALITIES FROM LORENTZ–FINSLER NORMS

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Abstract. We show that Lorentz-Finsler geometry offers a powerful tool in obtaining inequalities. With this aim, we first point out that a series of famous inequalities such as: the (weighted) arithmetic-geometric mean inequality, Aczél's, Popoviciu's and Bellman's inequalities, are all particular cases of a reverse Cauchy-Schwarz, respectively, of a reverse triangle inequality holding in Lorentz-Finsler geometry. Then, we use the same method to prove some completely new inequalities, including two refinements of Aczél's inequality.

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