Mathematical
Inequalities
& Applications
Volume 24, Number 2 (2021), 463–476
doi:10.7153/mia-2021-24-32

APPROXIMATE ω–ORTHOGONALITY AND ω–DERIVATION

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Abstract. We introduce the notion of approximate ω-orthogonality (referring to the numerical radius ω) and investigate its significant properties. Let $T,S \in \mathbb{B}(\mathscr{H})$ and $\varepsilon \in [0,1)$. We say that T is approximate ω-orthogonality to S and we write $T \perp_\omega^\varepsilon S$ if

$$\omega^2(T+\lambda S) \geqslant \omega^2(T) - 2\varepsilon\omega(T)\omega(\lambda S), \qquad \text{for all } \lambda \in \mathbb{C}.$$

We show that $T \perp_\omega^\varepsilon S$ if and only if $\inf_{\theta\in[0,2\pi)} D_\omega^\theta(T,S) \geqslant -\varepsilon\omega(T)\omega(S)$ in which $D_\omega^\theta(T,S) = \lim_{r\to 0^+} \dfrac{\omega^2(T+re^{i\theta}S)-\omega^2(T)}{2r}$; and this occurs if and only if for every $\theta \in [0,2\pi)$, there exists a sequence $\{x_n^\theta\}$ of unit vectors in $\mathscr{H}$ such that

$$\lim_{n\to\infty} |\langle Tx_n^\theta, x_n^\theta\rangle| = \omega(T) \text{ and } \lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle \overline{\langle Sx_n^\theta, x_n^\theta\rangle}\} \geqslant -\varepsilon\omega(T)\omega(S),$$

where $\omega(T)$ is the numerical radius of T.

Mathematics subject classification (2010): 47A12, 46B20, 46C05.
Keywords and phrases: Hilbert space, numerical radius, approximate ω-orthogonality, ω-derivation.

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