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AN EXTENSION OF A HARDY'S INEQUALITY AND ITS APPLICATIONS

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Abstract. Let $p>0$ and let $s\leqslant q$ be fixed parameters. The paper contains the proof of the sharp Hardy-type inequality

$$\sum_{n=1}^{\infty}\lambda_n\left(\frac{\lambda_1a_1+\lambda_2a_2+\ldots+\lambda_na_n}{\lambda_1+\lambda_2+\ldots+\lambda_n}\right)^{-p+s}a_n^{-s}$$
$$\leqslant\left(1+\frac{1}{p}\right)^{q-s}\sum_{n=1}^{\infty}\lambda_n\left(\frac{\lambda_1a_1+\lambda_2a_2+\ldots+\lambda_na_n}{\lambda_1+\lambda_2+\ldots+\lambda_n}\right)^{-p+q}a_n^{-q}$$

for any sequences $(a_n)_{n=1}^{\infty}$, $(\lambda_n)_{n=1}^{\infty}$ of positive numbers. The approach exploits dynamic programming-type techniques and rests on the identification of the explicit Bellman function associated with the estimate. As applications, related estimates for Hardy operators in $\mathbb{R}^d$ and harmonic maximal operators on probability spaces are obtained.

Mathematics subject classification (2020): 26D10, 26D15, 42B25.
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REFERENCES

[1] R. BELLMAN, *Dynamic programming*, Reprint of the 1957 edition. Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2010.
[2] D. L. BURKHOLDER, *Explorations in martingale theory and its applications*, École d'Été de Probabilités de Saint-Flour XIX—1989, 1–66, Lecture Notes in Math., **1464**, Springer, Berlin.
[3] M. CHRIST AND L. GRAFAKOS, *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc. **123** (1995), 1687–1693.
[4] D. CRUZ-URIBE, SFO, AND C. J. NEUGEBAUER, *The structure of the reverse Hölder classes*, Trans. Amer. Math. Soc. **347** (1995), 2941–2960.
[5] D. CRUZ-URIBE, *The minimal operator and the geometric maximal operator in* $\mathbb{R}^n$, Studia Math. **144**, 1 (2001), 1–37.
[6] D. CRUZ-URIBE, SFO, C. J. NEUGEBAUER, V. OLESEN, *Norm inequalities for the minimal and maximal operator, and differentiation of the integral*, Publ. Mat. **41** (1997), 577–604.
[7] G. HARDY, *Note on a theorem of Hilbert*, Math. Z. **6**, 3–4 (1920), 314–317.
[8] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd edn., Cambridge University Press, 1952.
[9] K. KNOPP, *Über Reihen mit positiven Gliedern*, J. London Math. Soc. **3** (1928), 205–211.
[10] A. KUFNER, B. OPIC, *Hardy-type inequalities*, Longman Scientific and Technical, Harlow, 1990.
[11] A. KUFNER AND L. E. PERSSON, *Weighted Inequalities of Hardy Type*, World Sci., River Edge, NJ, 2003.
[12] S. Z. LU, D. Y. YAN AND F. Y. ZHAO, *Sharp bounds for Hardy type operators on higher-dimensional product spaces*, J. Inequal. Appl., 2013, 1 (2013), 1–11.
[13] A. D. MELAS, *The Bellman functions of dyadic-like maximal operators and related inequalities*, Adv. Math. **192** (2005), 310–340.

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[14] A. D. MELAS AND E. N. NIKOLIDAKIS, *Sharp Lorentz estimates for dyadic-like maximal operators and related Bellman functions*, J. Geom. Anal. **27**, 4 (2017), 2644–2657.
[15] F. L. NAZAROV AND S. R. TREIL, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, St. Petersburg Math. J. **8** (1997), 721–824.
[16] E. NIKOLIDAKIS, T. STAVROPOULOS, *A refinement of a Hardy type inequality for negative exponents, and sharp applications to Muckenhoupt weights on* $\mathbb{R}$, Colloq. Math. **157** (2018), 295–308.
[17] A. OSĘKOWSKI, *Sharp martingale and semimartingale inequalities*, Monografie Matematyczne **72** (2012), Birkhäuser, 462 pp.
[18] A. OSĘKOWSKI, *Sharp* $L^{p,\infty} \to L^q$ *estimates for the dyadic-like maximal operators*, J. Fourier Anal. Appl. **20** (2014), 911–933.
[19] A. OSĘKOWSKI, *Sharp weak-type estimates for the dyadic-like maximal operators*, Taiwanese J. Math. **19** (2015), 1031–1050.
[20] A. OSĘKOWSKI, *Sharp logarithmic inequalities for Hardy operators*, Z. Anal. Anwend. **35** (2016), 1–20.