

A PICONE'S IDENTITY FOR THE $p(x)$ -LAPLACE EQUATIONS AND ITS APPLICATIONS

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Abstract. In this paper, we present a Picone identity for the $p(x)$ -Laplace equations and establish some applications of this formula, such as Caccioppoli inequalities, nonexistence of positive supersolutions, domain monotonicity property, uniqueness and simplicity of the first eigenvalue, Hardy type inequality, Barta-type inequality, a nonlinear system with singular nonlinearity and Sturmian comparison theorem for the $p(x)$ -Laplace equations.

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