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SIMPLE ACCURATE BALANCED ASYMPTOTIC APPROXIMATION OF WALLIS' RATIO USING EULER–BOOLE ALTERNATING SUMMATION

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Abstract. For integers $m \geqslant 1$ and $q \geqslant 2$, the Wallis ratio $w_m := \prod_{k=1}^{m} \frac{2k-1}{2k}$ is estimated as

$$\left| w_m - \frac{1}{\sqrt{m\pi}} \exp\left(- \sum_{i=1}^{\lfloor q/2 \rfloor} \frac{\left(1-4^{-i}\right) B_{2i}}{i(2i-1) \cdot m^{2i-1}} \right) \right| < \frac{1}{2} \exp\left(\rho_q^*(m)\right) \cdot \rho_q^*(m),$$

where B_k are the Bernoulli coefficients and

$$|\rho_q^*(m)| < \frac{\pi(q-2)!}{3(2m\pi)^{q-1}} < \frac{\pi}{3} \sqrt{\frac{2\pi}{q-1}} \cdot \left(\frac{q-1}{2me\pi} \right)^{q-1} \exp\left(\frac{1}{12(q-1)} \right).$$

Some accurate asymptotic estimates of π in terms of w_m are also given.

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