Mathematical
Inequalities
& Applications
Volume 24, Number 4 (2021), 1041–1064
doi:10.7153/mia-2021-24-73

THE QUASI–HYPERBOLICITY CONSTANT OF A METRIC SPACE

GEORGE DRAGOMIR* AND ANDREW NICAS

Abstract. We introduce the *quasi-hyperbolicity constant* of a metric space, a rough isometry invariant that measures how a metric space deviates from being Gromov hyperbolic. Gromov hyperbolicity, and also the lack thereof, has attracted considerable interest in the theory of networks. The quasi-hyperbolicity constant for an unbounded space lies in the closed interval $[1,2]$. It is equal to one for an unbounded Gromov hyperbolic space. For a CAT(0)-space, it is bounded from above by $\sqrt{2}$. The quasi-hyperbolicity constant of a Banach space that is at least two dimensional is bounded from below by $\sqrt{2}$, and for a non-trivial L_p-space it is exactly $\max\{2^{1/p}, 2^{1-1/p}\}$. If $0<\alpha<1$ then the quasi-hyperbolicity constant of the α-snowflake of any metric space is bounded from above by 2^{α}. We give an exact calculation in the case of the α-snowflake of the Euclidean real line.

Mathematics subject classification (2020): Primary 51K05; Secondary 46B20, 51F99, 51M10.
Keywords and phrases: δ-hyperbolic, quasi-isometry, rough isometry, Banach space, snowflake metric.

REFERENCES

[1] RÉKA ALBERT, BHASKAR DASGUPTA, AND NASIM MOBASHERI, *Topological implications of negative curvature for biological and social networks*, Phys. Rev. E **89** (2014), 032811.
[2] I. D. BERG AND I. G. NIKOLAEV, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicata **133** (2008), 195–218. MR 2390077
[3] LEONARD M. BLUMENTHAL, *Theory and applications of distance geometry*, Second edition, Chelsea Publishing Co., New York, 1970. MR 0268781
[4] MARTIN R. BRIDSON, *On the existence of flat planes in spaces of nonpositive curvature*, Proc. Amer. Math. Soc. **123** (1995), no. 1, 223–235. MR 1273477
[5] MARTIN R. BRIDSON AND ANDRÉ HAEFLIGER, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486
[6] PER ENFLO, *On the nonexistence of uniform homeomorphisms between L_p-spaces*, Ark. Mat. **8** (1969), 103–105. MR 0271719
[7] JI GAO AND KA-SING LAU, *On the geometry of spheres in normed linear spaces*, J. Austral. Math. Soc. Ser. A **48** (1990), no. 1, 101–112. MR 1026841
[8] M. GROMOV, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR 919829
[9] WILLIAM B. JOHNSON AND JORAM LINDENSTRAUSS, *Basic concepts in the geometry of Banach spaces*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 1–84. MR 1863689
[10] EDMOND JONCKHEERE AND POONSUK LOHSOONTHORN, *Geometry of network security*, Proceedings of the 2004 American Control Conference, Boston, MA, 2004, pp. 976–981.
[11] EDMOND JONCKHEERE, POONSUK LOHSOONTHORN, AND FARIBA ARIAEI, *Scaled Gromov four-point condition for network graph curvature computation*, Internet Math. **7** (2011), no. 3, 137–177. MR 2837770
[12] NAOTO KOMURO, KICHI-SUKE SAITO, AND RYOTARO TANAKA, *On the class of Banach spaces with James constant $\sqrt{2}$*, Math. Nachr. **289** (2016), no. 8–9, 1005–1020. MR 3512046

Paper MIA-24-73

[13] C. J. LENNARD, A. M. TONGE, AND A. WESTON, *Generalized roundness and negative type*, Michigan Math. J. **44** (1997), no. 1, 37–45. MR 1439667
[14] BOGDAN NICA AND JÁN ŠPAKULA, *Strong hyperbolicity*, Groups Geom. Dyn. **10** (2016), no. 3, 951–964. MR 3551185
[15] TAKASHI SATO, *An alternative proof of Berg and Nikolaev's characterization of* $\mathrm{CAT}(0)$ *-spaces via quadrilateral inequality*, Arch. Math. (Basel) **93** (2009), no. 5, 487–490. MR 2563595
[16] I. J. SCHOENBERG, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. of Math. (2) **38** (1937), no. 4, 787–793. MR 1503370
[17] YILUN SHANG, *Lack of Gromov-hyperbolicity in small-world networks*, Cent. Eur. J. Math. **10** (2012), no. 3, 1152–1158. MR 2902244
[18] NICOLE TOMCZAK-JAEGERMANN, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 38, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. MR 993774
[19] JUSSI VÄISÄLÄ, *Gromov hyperbolic spaces*, Expo. Math. **23** (2005), no. 3, 187–231. MR 2164775