BOUNDS FOR COMPLETELY MONOTONIC DEGREES OF REMAINDERS IN ASYMPTOTIC EXPANSIONS OF THE DIGAMMA FUNCTION

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Abstract. Motivated by several conjectures posed in the paper "F. Qi and A.-Q. Liu, *Completely monotonic degrees for a difference between the logarithmic and psi functions*, J. Comput. Appl. Math. **361** (2019), 366–371; https://doi.org/10.1016/j.cam.2019.05.001", the authors bound several completely monotonic degrees of the remainders in the asymptotic expansions of the logarithm of the gamma function and in the asymptotic expansions of the logarithm of the digamma function.

Mathematics subject classification (2020): Primary 33B15; Secondary 26A48, 41A60, 44A10.

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