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HARDY AND SOBOLEV INEQUALITIES FOR DOUBLE PHASE FUNCTIONALS ON THE UNIT BALL

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Abstract. We prove Hardy and Sobolev inequalities for double phase functionals $\Phi(x,t) = t^p + (b(x)t)^q$ on the unit ball $\mathbf{B}$, as a continuation of our paper [26], where $1 \leqslant p < q$, $b(\cdot)$ is non-negative and (radially) Hölder continuous of order $\theta \in (0,1]$. The Sobolev conjugate for Φ is given by $\Phi^*(x,t) = t^{p^*} + (b(x)t)^{q^*}$, where p^* and q^* denote the Sobolev exponent of p and q, respectively, that is, $1/p^* = 1/p - 1/n$ and $1/q^* = 1/q - 1/n$.

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REFERENCES

[1] P. BARONI, M. COLOMBO AND G. MINGIONE, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57** (2018), no. 2, Paper No. 62, 48 pp.
[2] P. BARONI, M. COLOMBO AND G. MINGIONE, *Non-autonomous functionals, borderline cases and related function classes*, St Petersburg Math. J. **27** (2016), 347–379.
[3] S.-S. BYUN, J. OH, *Regularity results for generalized double phase functionals*, Anal. PDE **13** (2020), no. 5, 1269–1300.
[4] S.-S. BYUN, S. RYU AND P. SHIN, *Calderon-Zygmund estimates for ω-minimizers of double phase variational problems*, Appl. Math. Letters **86** (2018), 256–263.
[5] M. COLOMBO AND G. MINGIONE, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 443–496.
[6] M. COLOMBO AND G. MINGIONE, *Bounded minimizers of double phase variational integrals*, Arch. Rat. Mech. Anal. **218** (2015), 219–273.
[7] C. DE FILIPPIS AND G. MINGIONE, *Lipschitz bounds and nonautonomous Integrals*, Arch. Ration. Mech. Anal. **242** (2021), no. 2, 973–1057.
[8] C. DE FILIPPIS AND J. OH, *Regularity for multi-phase variational problems*, J. Differential Equations **267** (2019), no. 3, 1631–1670.
[9] C. DE FILIPPIS AND G. PALATUCCI, *Hölder regularity for nonlocal double phase equations*, J. Differential Equations **267** (2019), no. 1, 547–586.
[10] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities. 2d ed. Cambridge*, at the University Press, 1952. xii+324 pp.
[11] P. HARJULEHTO AND P. HÄSTÖ, *Orlicz Spaces and Generalized Orlicz Spaces*, Lecture Notes in Mathematics, **2236**, Springer, Cham, 2019.
[12] P. HARJULEHTO, P. HÄSTÖ AND M. LEE, *Hölder continuity of quasiminimizers and ω-minimizers of functionals with generalized Orlicz growth*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **XXII** (2021), 549–582.
[13] P. HÄSTÖ, *The maximal operator on generalized Orlicz spaces*, J. Funct. Anal. **269** (2015), no. 12, 4038–4048; Corrigendum to "The maximal operator on generalized Orlicz spaces", J. Funct. Anal. **271** (2016), no. 1, 240–243.
[14] P. HÄSTÖ AND J. OK, *Maximal regularity for local minimizers of non-autonomous functionals*, to appear in J. Eur. Math. Soc.
[15] A. KUFNER AND B. OPIC, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, vol. 219, Longman Group UK Limited, London, 1990.

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[16] A. KUFNER, L.-E. PERSSON AND N. SAMKO, *Weighted inequalities of Hardy type*, second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

[17] F.-Y. MAEDA, Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Sobolev's inequality for double phase functionals with variable exponents*, Forum Math. **31** (2019), 517–527.

[18] F.-Y. MAEDA, Y. MIZUTA AND T. SHIMOMURA, *Growth properties of Musielak-Orlicz integral means for Riesz potentials*, Nonlinear Anal. **112** (2015), 69–83.

[19] P. MARCELLINI, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), no. 5, 391–409.

[20] P. MARCELLINI, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), no. 3, 267–284.

[21] V. G. MAZ'YA, *Sobolev Spaces*, Springer, Heidelberg, 2011.

[22] Y. MIZUTA, *Potential theory in Euclidean spaces*, Gakkōtosho, Tokyo, 1996.

[23] Y. MIZUTA, E. NAKAI, T. OHNO AND T. SHIMOMURA, *Campanato-Morrey spaces for the double phase functionals*, Rev. Mat. Complut. **33** (2020), 817–834.

[24] Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Sobolev's theorem for double phase functionals*, Math. Ineq. Appl. **23** (2020), 17–33.

[25] Y. MIZUTA AND T. SHIMOMURA, *Hardy-Sobolev inequalities in the half space for double phase functionals*, to appear in Rocky Mountain J. Math.

[26] Y. MIZUTA AND T. SHIMOMURA, *Hardy-Sobolev inequalities in the unit ball for double phase functionals*, J. Math. Anal. Appl. **501** (2021), 124133, 17 pp.

[27] Y. MIZUTA AND T. SHIMOMURA, *Boundary growth of Sobolev functions of monotone type for double phase functionals*, Ann. Fenn. Math. **47** (2022), 23–37.

[28] L.-E. PERSSON AND S. G. SAMKO, *A note on the best constants in some Hardy inequalities*, J. Math. Inequal. **9** (2015), no. 2, 437–447.

[29] M. A. RAGUSA AND A. TACHIKAWA, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), no. 1, 710–728.

[30] T. SHIMOMURA, *L^q-mean limits for Taylor's expansion of Riesz potentials of functions in Orlicz classes*, Hiroshima Math. J. **27** (1997), 159–175.

[31] V. V. ZHIKOV, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 675–710.