

FOURIER TRANSFORM OF VARIABLE ANISOTROPIC HARDY SPACES WITH APPLICATIONS TO HARDY–LITTLEWOOD INEQUALITIES

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Abstract. Let $p(\cdot) : \mathbb{R}^n \rightarrow (0, 1]$ be a variable exponent function satisfying the globally log-Hölder continuous condition and A a general expansive matrix on \mathbb{R}^n . Let $H_A^{p(\cdot)}(\mathbb{R}^n)$ be the variable anisotropic Hardy space associated with A defined via the radial maximal function. In this article, via the known atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$ and establishing two useful estimates on anisotropic variable atoms, the author shows that the Fourier transform \widehat{f} of $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ coincides with a continuous function F in the sense of tempered distributions, and F satisfies a pointwise inequality which contains a step function with respect to A as well as the Hardy space norm of f . As applications, the author also obtains a higher order convergence of the continuous function F at the origin. Finally, an analogue of the Hardy–Littlewood inequality in the variable anisotropic Hardy space setting is also presented. All these results are new even in the classical isotropic setting.

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