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SHARP NONLINEAR ESTIMATES FOR MULTIPLYING DERIVATIVES OF POSITIVE DEFINITE TENSOR FIELDS

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Abstract. The simple product formulae for derivatives of scalar functions raised to different powers are generalized for functions which take values in the set of symmetric positive definite matrices. These formulae are fundamental in derivation of various non-linear estimates, especially in the PDE theory. To get around the non-commutativity of the matrix and its derivative, we apply some well-known integral representation formulas and then we make an observation that the derivative of a matrix power is a logarithmically convex function with respect to the exponent. This is directly related to the validity of a seemingly simple inequality combining the integral averages and the inner product on matrices. The optimality of our results is illustrated on numerous examples.

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